Acta Numerica (2003), pp. 451–512
DOI: 10.1017/S0962492902000156

Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems

Eitan Tadmor*
Department of Mathematics,
Center for Scientific Computation
and Mathematical Modeling (CSCAMM)
and
Institute for Physical Science & Technology (IPST),
University of Maryland,
College Park, MD 20742, USA
E-mail: tadmor@cscamm.umd.edu

We study the entropy stability of difference approximations to nonlinear hyperbolic conservation laws, and related time-dependent problems governed by additional dissipative and dispersive forcing terms. We employ a comparison principle as the main tool for entropy stability analysis, comparing the entropy production of a given scheme against properly chosen *entropy-conservative* schemes.

To this end, we introduce general families of entropy-conservative schemes, interesting in their own right. The present treatment of such schemes extends our earlier recipe for construction of entropy-conservative schemes, introduced in Tadmor (1987*b*). The new families of entropy-conservative schemes offer two main advantages, namely, (i) their numerical fluxes admit an explicit, closed-form expression, and (ii) by a proper choice of their path of integration in phase space, we can distinguish between different families of waves within the same computational cell; in particular, entropy stability can be enforced on rarefactions while keeping the sharp resolution of shock discontinuities.

A comparison with the numerical viscosities associated with entropy-conservative schemes provides a useful framework for the construction and analysis of entropy-stable schemes. We employ this framework for a detailed study of entropy stability for a host of first- and second-order accurate schemes. The comparison approach yields a precise characterization of the entropy stability of semi-discrete schemes for both scalar problems and systems of equations.

* Research was supported by NSF grants DMS01-07917 and DMS01-07428 and by ONR grant N00014-91-J-1076.

We extend these results to fully discrete schemes. Here, spatial entropy dissipation is balanced by the entropy production due to time discretization with a sufficiently small time-step, satisfying a suitable CFL condition. Finally, we revisit the question of entropy stability for fully discrete schemes using a different approach based on *homotopy* arguments. We prove entropy stability under optimal CFL conditions.

CONTENTS

1. Introduction

We discuss the stability of difference approximations to conservation laws and related time-dependent problems. The related problems we have in mind are governed by additional dissipative and dispersive forcing terms. Our main focus, however, is devoted to nonlinear convection governed by hyperbolic systems of conservation laws. In the linear hyperbolic framework, L^2-stability is sought as a discrete analogue for the *a priori* energy estimates available in the differential set-up, *e.g.*, Richtmyer and Morton (1967) and Gustafsson, Kreiss and Oliger (1995); consult the recent *Acta Numerica* review by Kreiss and Lorenz (1998). In the present context of nonlinear problems dominated by nonlinear convection, we seek *entropy stability* as a discrete analogue for the corresponding statement in the differential set-up. The prototype one-dimensional problem consists of systems of conservation laws, $\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x = 0$. A distinctive feature of this problem is the spontaneous formation of shock discontinuities. The entropy condition plays a decisive role in the theory and numerics of such problems (Lax 1972, Smoller 1983, Dafermos 2000). It requires $\mathbf{u}$ to satisfy the additional inequality, $U(\mathbf{u})_t + F(\mathbf{u})_x \leq 0$, for all admissible entropy pairs $(U(\mathbf{u}), F(\mathbf{u}))$. It follows that the total amount of entropy, $\int U(\mathbf{u}(\cdot,t)\,\mathrm{d}x$, does not increase in time. This is a generalization of the (weighted) L^2-energy bound encountered in the linear case. The possibility of strict inequality reflects entropy decay due to concentration along shock discontinuities.

We consider difference approximations of the general conservative form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) + \frac{\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}}{\Delta x_\nu} = 0.$$

Here $\mathbf{u}_\nu(t)$ is the numerical solution computed at discrete grid lines (x_ν, t), and $\mathbf{f}_{\nu+\frac{1}{2}} \sim \mathbf{f}$ is a numerical flux based on a stencil of neighbouring grid values, $\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p}$. We enquire when such schemes are entropy-stable in the sense of satisfying the corresponding discrete entropy inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}U(\mathbf{u}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x_\nu} \leq 0.$$

So far we have specified semi-discrete schemes based on spatial differencing. We will address the question of entropy stability for the semi-discrete as well as the fully discrete case, taking into account additional temporal discretization. The extension to the multidimensional set-up and a host of related problems with additional dissipative and dispersive terms can be handled in a straightforward manner.

We distinguish between three main tools of the trade in the analysis of entropy stability: comparison arguments, a homotopy approach and kinetic formulations. We will discuss the first two and refer the reader to Bouchut (2002), Makridakis and Perthame (2003) and the references therein for recent contributions regarding the third. Most of our discussion will be devoted to the main approach, based on a *comparison principle*: we compare the amount of entropy dissipation produced by a given scheme against a properly chosen entropy-stable reference. The entropy stability of solutions to monotone schemes, for example, Harten, Hyman and Lax (1976), is carried out by a comparison with the (entropy-stable) constant solution (Crandall and Majda 1980). The class of entropy-stable E-schemes (Osher 1984) is characterized by having more numerical viscosity than the entropy-stable Godunov scheme (Tadmor 1984*b*). And we mention in passing the kinetic approach presented in Makridakis and Perthame (2003), which is based on comparison of the corresponding pseudo-Maxwellians.

In Tadmor (1987*b*), the question of entropy stability was addressed by the construction of certain *entropy-conservative* schemes, interesting for their own sake. We begin, in Section 3, with the construction of these entropy-conservative schemes. There are two main ingredients: (i) the use of entropy variables, outlined in Section 2, and (ii) the choice of certain paths of integration in phase space of these entropy variables. In the scalar case, the numerical fluxes are path-independent, and entropy-conservative schemes are unique (for a given entropy pair). In Section 4 we study a host of instructive scalar examples whose entropy stability is verified by comparison with entropy-conservative ones. These include the first-order Engquist–Osher, the optimality of the Godunov scheme and the second-order Lax–Wendroff, as well as other centred schemes. In Section 5 we turn our attention to systems. Here we revisit the construction of entropy-conservative schemes in terms of numerical fluxes which are integrated along straight line paths in phase space. A comparison of numerical viscosities provides a detailed study of entropy stability for Rusanov, Lax–Friedrichs and the family of Roe-type schemes, as well as second-order extensions. In Section 6 we present the general framework, introducing new families of entropy-conservative fluxes subject to the choice of path of integration in phase space. These new entropy-conservative schemes offer two main advantages: (i) their numerical fluxes admit an explicit, closed-form expression, and, more importantly, (ii) by a proper choice of the path of integration (aligned with the eigen-directions of $\mathbf{f_u}$), one can distinguish between different families of waves within the same cell, $[x_\nu, x_{\nu+1}]$. In particular, entropy stability can be enforced on rarefactions while keeping the sharp resolution of shock discontinuities. In Section 7 we extend our discussion to fully discrete schemes,

$$\mathbf{u}_\nu^{n+1} \equiv \mathbf{u}_\nu(t^n + \Delta t) = \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x_\nu}\Big[\mathbf{f}_{\nu+\frac{1}{2}}(\overline{\mathbf{u}}^{n+\frac{1}{2}}) - \mathbf{f}_{\nu-\frac{1}{2}}(\overline{\mathbf{u}}^{n+\frac{1}{2}})\Big].$$

There are three prototype examples. In the fully implicit case where we set $\overline{\mathbf{u}}^{n+\frac{1}{2}} := \mathbf{u}^{n+1}$, additional entropy dissipation is introduced by the time discretization and hence this implicit backward Euler scheme is entropy-stable whenever the semi-discrete scheme is. In the case of Crank–Nicolson time discretization, a proper (possibly nonlinear) choice of intermediate values $\overline{\mathbf{u}}^{n+\frac{1}{2}}$ inherits the same unconditional entropy stability properties of the semi-discrete problem associated with the numerical flux $\mathbf{f}_{\nu+\frac{1}{2}}$; finally, the fully explicit case, $\overline{\mathbf{u}}^{n+\frac{1}{2}} := \mathbf{u}^n$, yields entropy production which needs to be balanced by entropy dissipation on the spatial part. This balance is achieved for a mesh ratio satisfying a suitable Courant–Friedrichs–Lewy (CFL) condition, $\frac{\Delta t}{\Delta x}\|\mathbf{f_u}\| \leq \text{Const}$.

In Section 8 we revisit the question of entropy stability using a completely different approach, based on *homotopy* arguments. The results apply to semi- and fully discrete approximations of scalar and systems of conservation laws. We prove the entropy stability for a large class of first-order schemes, this time under an optimal CFL condition. For second-order scalar extensions we refer to Nessyahu and Tadmor (1990, Appendix). The homotopy argument was introduced by Lax (1971) in the context of the Lax–Friedrichs scheme.

The entropy stability study is based on comparison with entropy-conservative schemes. The entropy-stable schemes discussed so far were limited by the use of second-order accurate entropy-conservative schemes as a reference for a comparison. We conclude, in Section 9, with higher-order extensions. We recast the original entropy-conservative schemes in their piecewise linear finite element formulation (Tadmor 1986*b*). Higher orders with larger stencils follow from piecewise polynomials of higher degrees. A general framework for such high-order entropy-conservative schemes was recently introduced in LeFloch, Mercier and Rohde (2002), and should serve as the starting point for the corresponding higher-order entropy stability analysis.

Our discussion on entropy stability theory is tied to several topics which we were unable to explore in the present framework, and we conclude this Introduction by mentioning a few of the items that were left out.

- *Entropy-conservative schemes* play an essential role in our discussion below as the main reference for calibrating entropy stability. Entropy-conservative schemes are interesting in their own right in the context of zero dispersion limits, and completely integrable systems (consult, for example, Lax, Levermore and Venakidis (1993) and Deift and McLaughlin (1998)), with much recent renewed interest (Abramov, Kovačič and Majda 2003, Abramov and Majda 2002). Entropy-conservative schemes are also sought in the context of energy conservation for long-term shock-free integration: for example, Arakawa (1966). Let us mention the related class of completely conservative schemes developed by the school of A. A. Samarskii and co-workers: consult, for example, Moskalkov (1980) and the references therein.
- *Entropy stability* serves as an essential guideline in the design of new computationally reliable difference schemes. Much of our discussion below is devoted to the development of a general framework for proving the entropy stability of such schemes. As an alternative approach, we mention the design of entropy *corrections* for existing schemes. For early numerical simulations with entropy corrections along these lines, we refer to Khalfallah and Lerat (1988) and Kaddouri (1993), for example. The new class of entropy-conservative/entropy-stable schemes explored in Section 6 offers a challenging new set-up for revisiting numerical simulations with entropy corrected schemes.
- *Entropy variables* are essential for symmetrization, and hence for the sense of ordering required for the comparison approach in verifying entropy stability. Entropy variables are essential for the weak finite element formulation as briefly outlined in Section 9. We refer to the streamline diffusion of Hughes, Johnson and collaborators (Hughes, Franca and Mallet 1986, Johnson and Szepessy 1986) as an example of a successful class of entropy variables-based finite element methods (FEM) for treating convection-dominated problems.

- *Compensated compactness*. Quantifying the amount of entropy dissipation (consult Corollary 5.1 below) enables us to convert the entropy stability statement into a convergence proof by compensated compactness arguments (Tartar 1975, DiPerna 1983, Chen 2000). This methodology was applied to different classes of discrete methods: for instance, FEM streamline diffusion (Johnson, Szepessy and Hansbo 1990), the spectral viscosity method (Tadmor 1989), and multidimensional finite volume methods (LeVeque 2002). The present framework should pave the way for a systematic development of a convergence theory for a large class of entropy-stable finite difference approximations (for scalar and 2×2 systems).
- *Nonclassical shocks* and a host of nonlinear phenomena are governed by a borderline balance between dissipative and dispersive forces: we refer, for example, to the recent phase transitions studies of LeFloch and co-workers (LeFloch 2002). The numerical simulation in those regimes becomes possible by carefully tuning the amount of entropy dissipation/dispersion added to the entropy-conservative schemes.
- *Boundary conditions*. Once the entropy-conservative schemes are introduced, the question of entropy stability is answered by summation by parts, carried out in phase space of entropy variables. This reveals the skew-symmetry of the spatial operators (Tadmor 1984*a*), while retaining the conservative form. Consequently, summation by parts along these lines should in principle enable us to treat the question of entropy stability in the presence of boundaries: consult Olsson (1995), for example.

2. The entropy variables

We consider systems of conservation laws of the form

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{2.1}$$

where $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_N(\mathbf{u}))^\top$ are smooth flux functions of the N-vector of conservative variables[1] $\mathbf{u}(x,t) = (u_1(x,t), \ldots, u_N(x,t))^\top$. We assume that system (2.1) is equipped with a convex *entropy function*, $U(\mathbf{u})$, such that

$$U_{\mathbf{uu}}A = [U_{\mathbf{uu}}A]^\top, \quad A(\mathbf{u}) := \mathbf{f}_{\mathbf{u}}(\mathbf{u}). \tag{2.2}$$

Thus, the Hessian of an entropy function symmetrizes the system (2.1) upon multiplication 'on the left' (Friedrichs and Lax 1971). An alternative

[1] Here and below, scalars are distinguished from vectors, which are denoted by **bold** letters.

procedure, which respects both strong and weak solutions of (2.1), is to symmetrize 'on the right', where (2.2) is replaced by the equivalent statement

$$A(U_{\mathbf{uu}})^{-1} = \left[A(U_{\mathbf{uu}})^{-1}\right]^\top. \tag{2.3}$$

To this end, Mock (1980) (see also Godunov (1961)) suggested the following procedure. Define the *entropy variables*

$$\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \nabla_{\mathbf{u}} U(\mathbf{u}). \tag{2.4}$$

Thanks to the convexity of $U(\mathbf{u})$, the mapping $\mathbf{u} \to \mathbf{v}$ is one-to-one and hence we can make the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$, which puts the system (2.1) into its equivalent symmetric form

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{v}) + \frac{\partial}{\partial x}\mathbf{g}(\mathbf{v}) = 0, \quad \mathbf{g}(\mathbf{v}) := \mathbf{f}(\mathbf{u}(\mathbf{v})). \tag{2.5}$$

Here, $\mathbf{u}(\cdot)$ and $\mathbf{g}(\cdot)$ become the temporal and spatial fluxes in the independent entropy variables, $\mathbf{v}$, and the system (2.5) is symmetric in the sense that the Jacobians of these fluxes are, namely

$$H(\mathbf{v}) := \mathbf{u}_{\mathbf{v}}(\mathbf{v}) = H^\top(\mathbf{v}) > 0 \quad \text{and} \quad B(\mathbf{v}) := \mathbf{g}_{\mathbf{v}}(\mathbf{v}) = B^\top(\mathbf{v}). \tag{2.6}$$

Indeed, (2.2) holds if and only if there exists an entropy flux function, $F = F(\mathbf{u})$, such that the following compatibility relation holds:

$$U_{\mathbf{u}}^\top \mathbf{f}_{\mathbf{u}} = F_{\mathbf{u}}^\top. \tag{2.7}$$

Consequently, we have

$$\mathbf{u}(\mathbf{v}) = \nabla_{\mathbf{v}}\phi(\mathbf{v}), \quad \phi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{u}(\mathbf{v})\rangle - U(\mathbf{u}(\mathbf{v})) \tag{2.8}$$

$$\mathbf{g}(\mathbf{v}) = \nabla_{\mathbf{v}}\psi(\mathbf{v}), \quad \psi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{g}(\mathbf{v})\rangle - F(\mathbf{u}(\mathbf{v})), \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. Hence the Jacobians $H(\mathbf{v})$ and $B(\mathbf{v})$ in (2.6) are symmetric, being the Hessians of $\phi(\mathbf{v})$ and $\psi(\mathbf{v})$. The latter, so-called potential functions, $\phi(\mathbf{v})$ and $\psi(\mathbf{v})$, are significant tools in our discussion below. Observe that the symmetry of $B = AH$ amounts to the symmetrization 'on the right' indicated in (2.3).

Entropy functions play an important role in the stability theory of PDEs dominated by the nonlinear convection of the type (2.1). We provide below a brief overview and refer the reader to a detailed account in Volpert (1967), Kružkov (1970), Friedrichs and Lax (1971), Lax (1972), Tartar (1975), DiPerna (1983), Smoller (1983), Majda (1984), Serre (1999), Dafermos (2000) and LeFloch (2002). We first recall that 'physically relevant' solutions of (2.1), are those arising as vanishing viscosity limits, $\mathbf{u} = \lim_{\epsilon \downarrow 0} \mathbf{u}^\epsilon$, where

$$\mathbf{u}_t^\epsilon + \mathbf{f}(\mathbf{u}^\epsilon)_x = \epsilon(P\mathbf{u}_x^\epsilon)_x. \tag{2.10}$$

Here $P = P(\mathbf{u}, \mathbf{u}_x)$ is any admissible viscosity matrix which is H-symmetric (compare (2.3)), that is,

$$PH = \big[PH\big]^\top \geq 0, \quad H = (U_{\mathbf{uu}})^{-1}, \tag{2.11}$$

so that integration of (2.10) against $U_{\mathbf{u}}^\top$ yields

$$\begin{aligned}\frac{\partial}{\partial t}U(\mathbf{u}^\epsilon) + \frac{\partial}{\partial x}F(\mathbf{u}^\epsilon) &= \left\langle U_{\mathbf{u}}^\top, \frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u})\right\rangle \\ &= -\epsilon\langle U_{\mathbf{uu}}\mathbf{u}_x^\epsilon, P\mathbf{u}_x^\epsilon\rangle \\ &= -\epsilon\big\langle (H^{-1}\mathbf{u}_x^\epsilon), PH(H^{-1}\mathbf{u}_x^\epsilon)\big\rangle \leq 0. \end{aligned} \tag{2.12}$$

Passing to the limit we obtain the *entropy inequality*

$$\frac{\partial}{\partial t}U(\mathbf{u}) + \frac{\partial}{\partial x}F(\mathbf{u}) \leq 0. \tag{2.13}$$

The passage to the limit on the right of (2.12) is understood weakly, in the sense of measures; the passage inside the nonlinear terms on the left, however, requires strong limits: consult the recent breakthrough of Bianchini and Bressan (Bianchini and Bressan 2003, Bressan 2003). The possibility of a strictly negative measure on the left of (2.13) is due to concentration of entropy dissipation along shock discontinuities on the right of (2.12).

The entropy inequality (2.13) is necessary in order to single out a unique, 'physically relevant' solution among the possibly many weak solutions of (2.1). In this context it is important whether (2.1) is endowed with a sufficiently 'rich' family of entropy pairs, (U, F): consult, for example, Serre (1991). How 'rich' is the family of such entropy functions? In the scalar case, $N = 1$, scalar Jacobians are symmetric and hence *every* convex U serves as an entropy function. This is the starting point for the L^1-stability theory of Kružkov (1970) for general scalar equations; we postpone this discussion to the end of this section. If the $N \times N$ system happens to be symmetric to begin with, then we can use the identity as a symmetrizer in (2.2), $U_{\mathbf{uu}} = I_N$, and hence the usual 'energy', $U(\mathbf{u}) = |\mathbf{u}|^2/2$, is an entropy function (Godunov 1961). In this case, integration of (2.13) yields the entropy bound

$$\int_x U(\mathbf{u}(x,t))\,\mathrm{d}x \leq \int_x U(\mathbf{u}(x,0))\,\mathrm{d}x, \tag{2.14}$$

which is the usual L^2-stability statement familiar from the linear theory of symmetric hyperbolic systems. Thus, entropy stability could be viewed as a nonlinear extension of the L^2 linear stability set-up to general, non-symmetric $N \times N$ systems. For 2×2 systems, the symmetrizing requirement from an entropy function, (2.2), amounts to a second-order linear hyperbolic equation and Lax (1971) has shown how to construct a family of entropy functions in this case. For general $N \geq 3$ equations, (2.2) is over-determined.

Nevertheless, most physically relevant systems are equipped with (at least one) entropy pair. The canonical example is of course the following one.

Example 2.1. (Euler equations) We consider entropy solutions, $\mathbf{u} = (\rho, m, E)^\top$ of the Euler equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ m\\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}m\\ qm+p\\ q(E+p)\end{bmatrix} = 0. \tag{2.15}$$

These equations govern inviscid polytropic gas dynamics, asserting the conservation of the density ρ, the momentum m, and the total energy E. Here q and p are, respectively, the velocity $q := \frac{m}{\rho}$ and the pressure $p = (\gamma-1)\cdot\left[E - \frac{m^2}{2\rho}\right]$ (where γ is the adiabatic exponent). Harten has shown that this system of equations is equipped with a family of entropy pairs, (U, F). These pairs take the form

$$U(\mathbf{u}) = -\rho h(S), \quad F(\mathbf{u}) = -mh(S) \tag{2.16}$$

(Harten 1983*b*; consult also Tadmor (1986*a*)). Here S stands for the non-dimensional specific entropy

$$S = \ell n(p\rho^{-\gamma}), \tag{2.17}$$

and $h = h(S)$ is any scalar function satisfying

$$h' - \gamma h'' > 0, \quad h' > 0, \tag{2.18}$$

so that the requirement for $U(\mathbf{u})$ to be convex is met (Harten 1983*b*). The corresponding entropy variables are given by

$$\mathbf{v} \equiv \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = (1-\gamma)\cdot\frac{h'(S)}{p}\cdot\begin{bmatrix}E + \frac{p}{\gamma-1}\left(\frac{h(S)}{h'(S)} - \gamma - 1\right)\\ -m\\ \rho\end{bmatrix}, \tag{2.19}$$

with the corresponding potential pairs, $(\phi, \psi) = (\gamma-1)h'(S)(\rho, m)$. A particularly convenient form to work with is determined by $h(S) = \frac{\gamma+1}{\gamma-1}\cdot \mathrm{e}^{\frac{S}{\gamma+1}}$. With this choice we find the entropy pair

$$U(\mathbf{u}) = \frac{\gamma+1}{1-\gamma}\cdot(\rho p)^{\frac{1}{\gamma+1}}, \quad F(\mathbf{u}) = \frac{\gamma+1}{1-\gamma}q\cdot(\rho p)^{\frac{1}{\gamma+1}}, \tag{2.20}$$

with the corresponding entropy variables, $\mathbf{v} = \mathbf{v}(\mathbf{u})$, given by

$$\mathbf{v} \equiv \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = -(\rho p)^{-\frac{\gamma}{\gamma+1}}\cdot\begin{bmatrix}E\\ -m\\ \rho\end{bmatrix}. \tag{2.21}$$

The inverse mapping, $\mathbf{v} \to \mathbf{u}$, is easily obtained as

$$\mathbf{u} = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix} = -(\rho p)^{\frac{\gamma}{\gamma+1}} \begin{bmatrix} v_3 \\ -v_2 \\ v_1 \end{bmatrix}, \tag{2.22}$$

where

$$\rho p = \left[(\gamma-1)\left(v_1 v_3 - \frac{v_2^2}{2}\right)\right]^{\frac{1+\gamma}{1-\gamma}}. \tag{2.23}$$

Godunov has studied the special choice $h(S) = S$, which leads to the canonical 'physical' entropy pair

$$U(\mathbf{u}) = -\rho S, \quad F(\mathbf{u}) = -mS. \tag{2.24}$$

Expressed in terms of the absolute temperature, T, the entropy variables in this case read

$$\mathbf{v} \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -\frac{c_v}{T} \begin{bmatrix} T(S-\gamma) + \frac{q^2}{2} \\ -q \\ 1 \end{bmatrix}, \qquad T := (\gamma-1)\cdot c_v \cdot \frac{p}{\rho}, \tag{2.25}$$

and the inverse mapping $\mathbf{v} \to \mathbf{u}$ can be found in Harten (1983*b*). We conclude this example with several remarks.

(1) The Euler equations (2.15) provide us with an example which shows how the 'richness' of the entropy pairs can be used for a stability statement: using the one-parameter family[2] $(-\rho(S-c)^-, -m(S-c)^-)$, which is admissible by (2.18), we obtain a minimum entropy principle, $S(x,t) \ge \min_y S(y,0)$ (consult Tadmor (1986*a*)).

(2) We note that the family of admissible entropy pairs, (2.16), (2.17), (2.18), becomes smaller once we seek further symmetrization of the viscous Navier–Stokes terms along the lines of (2.11) (consult Hughes *et al.* (1986)).

(3) Finally, we call attention to the fact that, with the particular choice of entropy pair $(U,F) = (-\rho S, -mS)$ (consult (2.24), (2.25)), the corresponding potential pair (ϕ, ψ) turns out to be the density and momentum components of the flow, $(\phi(\mathbf{v}), \psi(\mathbf{v})) = (\gamma-1)(\rho, m)$. Hence, in view of (2.8), (2.9), Euler equations can be rewritten in the intriguing form

$$\frac{\partial}{\partial t}[\nabla_{\mathbf{v}}\rho] + \frac{\partial}{\partial x}[\nabla_{\mathbf{v}} m] = 0. \tag{2.26}$$

[2] The superscript $^+$ (respectively $^-$) denotes the positive (respectively negative) part of the indicated scalar.

We close the section with the promised discussion on the entropy stability of the scalar case. We start with the following result, extending the penetrating scalar arguments of Kružkov (1970), which demonstrates how the 'richness' of the family of entropy pairs is converted into a stability statement.

Theorem 2.2. (Tadmor 1997, Theorem 2.1) Assume the system (2.1) is endowed with an N-parameter family of entropy pairs, $(U(\mathbf{u};\mathbf{c}),\ F(\mathbf{u};\mathbf{c}))$, $\mathbf{c}\in\mathbb{R}^N$, satisfying the symmetry property

$$U(\mathbf{u};\mathbf{c}) = U(\mathbf{c};\mathbf{u}), \quad F(\mathbf{u};\mathbf{c}) = F(\mathbf{c};\mathbf{u}). \tag{2.27}$$

Let $\mathbf{u}^1, \mathbf{u}^2$ be two entropy solutions of (2.1). Then the following *a priori* estimate holds:

$$\int_x U(\mathbf{u}^1(x,t);\mathbf{u}^2(x,t))\,\mathrm{d}x \le \int_x U(\mathbf{u}^1(x,0);\mathbf{u}^2(x,0))\,\mathrm{d}x. \tag{2.28}$$

Sketch of proof. Let $\mathbf{u}^1(x,t)$ be an entropy solution of (2.1) satisfying the entropy inequality (2.13). We employ the latter with the entropy pair $(U(\mathbf{u};\mathbf{c}), F(\mathbf{u};\mathbf{c}))$ parametrized with $\mathbf{c}=\mathbf{u}^2(y,\tau)$. This tells us that $\mathbf{u}^1(x,t)$ satisfies

$$\partial_t U(\mathbf{u}^1(x,t);\mathbf{u}^2(y,\tau)) + \partial_x F(\mathbf{u}^1(x,t);\mathbf{u}^2(y,\tau)) \le 0. \tag{2.29}$$

Let φ_δ denote a symmetric C_0^∞ unit mass mollifier that converges to Dirac mass in $\mathbb{R}$ as $\delta\downarrow 0$; set $\phi_\delta(x-y,t-\tau) := \varphi_\delta(\frac{x-y}{2})\varphi_\delta(\frac{t-\tau}{2})$ as an approximate Dirac mass in $\mathbb{R}\times\mathbb{R}^+$. 'Multiplication' of the (distributional) entropy inequality (2.13) by $\phi_\delta(x-y,t-\tau)$ yields

$$\begin{aligned}&\partial_t(\phi_\delta U(\mathbf{u}^1;\mathbf{u}^2)) + \partial_x(\phi_\delta F(\mathbf{u}^1;\mathbf{u}^2))\\ &\qquad \le (\partial_t\phi_\delta)U(\mathbf{u}^1;\mathbf{u}^2) + (\partial_x\phi_\delta)F(\mathbf{u}^1;\mathbf{u}^2).\end{aligned} \tag{2.30}$$

A dual manipulation, this time with (y,τ) as the primary integration variables of $\mathbf{u}^2(y,\tau)$ and (x,t) parametrizing $\mathbf{c}=\mathbf{u}^1(x,t)$, yields

$$\begin{aligned}&\partial_\tau(\phi_\delta U(\mathbf{u}^2;\mathbf{u}^1)) + \partial_y(\phi_\delta F(\mathbf{u}^2;\mathbf{u}^1))\\ &\qquad \le (\partial_\tau\phi_\delta)U(\mathbf{u}^2;\mathbf{u}^1) + (\partial_y\phi_\delta)F(\mathbf{u}^2;\mathbf{u}^1).\end{aligned} \tag{2.31}$$

We now add the last two inequalities: by the symmetry property (2.27), the sum of the right-hand sides of (2.30) and (2.31) vanishes; whereas by sending δ to zero, the sum of the left-hand sides of (2.30) and (2.31) amounts to

$$\partial_t U(\mathbf{u}^1(x,t);\mathbf{u}^2(x,t)) + \partial_x F(\mathbf{u}^1(x,t);\mathbf{u}^2(x,t)) \le 0.$$

The result follows by spatial integration. ∎

Let us point out that the elegance of the last result is confronted with the difficulty of satisfying the symmetry property (2.27). Thus, for example, $N \times N$ symmetric hyperbolic systems are endowed with the N-parameter family of entropies $U(\mathbf{u}, \mathbf{c}) = |\mathbf{u} - \mathbf{c}|^2/2$, but (2.27) fails for the corresponding entropy fluxes, $F(\mathbf{u}, \mathbf{c}) = \langle \mathbf{u} - \mathbf{c}, f(\mathbf{u}) \rangle - \int^{\mathbf{u}} \langle \mathbf{f}(\mathbf{w}), \mathrm{d}\mathbf{w} \rangle$. The favourable situation occurs in the scalar case where each convex U serves as an entropy function. In particular, Kružkov (1970) set the one-parameter family

$$U(u) = |u - c|, \qquad F(u) = \operatorname{sgn}(u - c)(f(u) - f(c)).$$

The symmetry requirement (2.27) holds and (2.28) leads to the following L^1-stability estimate.

Corollary 2.3. (Kružkov 1970) If u^1, u^2 are two entropy solutions of the scalar conservation law (2.1) subject to L^1 initial data, then

$$\|u^2(\cdot, t) - u^1(\cdot, t)\|_{L^1(x)} \leq \|u^2(\cdot, 0) - u^1(\cdot, 0)\|_{L^1(x)}. \tag{2.32}$$

Thus there exists a unique (entropy) solution operator associated with the scalar conservation law (2.1), $\mathcal{S}(t) : u(\cdot, 0) \mapsto u(\cdot, t)$, which is conservative and, according to Corollary 2.3, is also L^1-contractive, and hence by the Crandall–Tartar lemma (Crandall and Tartar 1980), $\mathcal{S}$ is order-preserving, $u^2(\cdot, 0) \geq u^1(\cdot, 0) \Longrightarrow \mathcal{S}(t)u^2(\cdot, 0) \geq \mathcal{S}(t)u^1(\cdot, 0)$. There is a parallel discrete theory for so-called *monotone schemes* which respect a similar discrete property of order preserving. The entropy stability of such schemes goes back to the pioneering work of Harten *et al.* (1976). We will not be able to expand on the details in the limited framework of this review, but let us mention the elegant approach of Crandall and Majda (1980), which clarified the entropy stability of monotone schemes in terms of a *comparison* with the constant solution. Sanders (1983) generalized the result to variable grids and we refer to Godlewski and Raviart (1996), Kröner (1997), Tadmor (1998) and LeVeque (2002) and the references therein for a series of later works, with particular emphasis on multidimensional extensions. Monotone schemes are at most first-order accurate (Harten *et al.* 1976); indeed, being entropy-stable with respect to all convex entropies, monotone schemes are necessarily limited to first-order accuracy (Osher and Tadmor 1988). This limitation led to systematic development of high-resolution schemes which circumvent this first-order limitation. For a brief overview of the convergence analysis of such schemes, we refer to Tadmor (1998). Our discussion below focuses on the question of entropy stability of such first-order as well as higher-order resolution schemes, in the context of both scalar and systems of conservation laws.

3. Entropy-conservative and entropy-stable schemes

We consider semi-discrete conservative schemes of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\Big], \tag{3.1}$$

serving as consistent approximations to (2.1). Here, $\mathbf{u}_\nu(t)$ denotes the discrete solution along the grid line (x_ν, t) with $\Delta x_\nu := \frac{1}{2}(x_{\nu+1} - x_{\nu-1})$ being the variable meshsize, and $\mathbf{f}_{\nu+\frac{1}{2}}$ being the Lipschitz-continuous numerical flux consistent with the differential flux, that is,

$$\mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p}), \quad \mathbf{f}(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) \equiv \mathbf{f}(\mathbf{u}). \tag{3.2}$$

The numerical flux, $\mathbf{f}(\cdot, \cdot, \ldots, \cdot)$, involves a stencil of $2p$ neighbouring grid values, and as such could be clearly distinguished from the (same notation of) the differential flux, $\mathbf{f}(\cdot)$. The difference schemes (3.1) and (3.2) are conservative in the sense of Lax and Wendroff (1960), namely, the change of total mass,

$$\sum_{\nu=-L}^{R} \mathbf{u}_\nu(t)\Delta x_\nu,$$

is solely due to the flux through the local neighbourhoods of the *arbitrary* boundaries at x_{-L} and x_R.

We are concerned here with the question of *entropy stability* of such schemes. To this end, let (U, F) be an entropy pair associated with the system (2.1). We ask whether the scheme (3.1) is *entropy-stable* with respect to such a pair, in the sense of satisfying a discrete entropy inequality analogous to (2.13), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] \le 0. \tag{3.3}$$

Here, $F_{\nu+\frac{1}{2}}$ is a consistent numerical entropy flux

$$F_{\nu+\frac{1}{2}} = F(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p}), \quad F(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = F(\mathbf{u}). \tag{3.4}$$

If, in particular, equality holds in (3.3), we say that the scheme (3.1) is *entropy-conservative.*

The answer to this question of entropy stability provided in Tadmor (1987*b*) consists of two main ingredients: (i) the use of the entropy variables and (ii) the comparison with appropriate *entropy-conservative* schemes. We conclude this section with a brief overview.

By making the changes of variables $\mathbf{u}_\nu = \mathbf{u}(\mathbf{v}_\nu)$, the scheme (3.1) recasts into the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\Big], \quad \mathbf{u}_\nu(t) = \mathbf{u}(\mathbf{v}_\nu(t)), \tag{3.5}$$

with a numerical flux

$$\mathbf{g}_{\nu+\frac{1}{2}} = \mathbf{g}(\mathbf{v}_{\nu-p+1}, \dots, \mathbf{v}_{\nu+p}) := \mathbf{f}(\mathbf{u}(\mathbf{v}_{\nu-p+1}), \dots, \mathbf{u}(\mathbf{v}_{\nu+p})), \tag{3.6}$$

consistent with the differential flux, that is,

$$\mathbf{g}(\mathbf{v}, \mathbf{v}, \dots, \mathbf{v}) = \mathbf{g}(\mathbf{v}) \equiv \mathbf{f}(\mathbf{u}(\mathbf{v})). \tag{3.7}$$

Define

$$F_{\nu+\frac{1}{2}} := \frac{1}{2}\Big\langle [\mathbf{v}_\nu + \mathbf{v}_{\nu+1}], \mathbf{g}_{\nu+\frac{1}{2}} \Big\rangle - \frac{1}{2}\big[\psi(\mathbf{v}_\nu) + \psi(\mathbf{v}_{\nu+1})\big]. \tag{3.8}$$

Then the following identity holds:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] \qquad (3.9)\\ &\quad = \frac{1}{2}\Big[\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}_{\nu+\frac{1}{2}} \Big\rangle - \Delta\psi_{\nu+\frac{1}{2}}\Big] + \frac{1}{2}\Big[\Big\langle \Delta\mathbf{v}_{\nu-\frac{1}{2}}, \mathbf{g}_{\nu-\frac{1}{2}} \Big\rangle - \Delta\psi_{\nu-\frac{1}{2}}\Big] \end{aligned}$$

(Tadmor 1987*b*, Section 4; see also Osher (1984)). Here $\Delta\psi_{\nu+\frac{1}{2}} := \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)$ denotes the difference of entropy flux potential, (2.9), of two neighbouring grid values $\mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1}$. Thanks to (2.9), $F_{\nu+\frac{1}{2}}$ is a consistent entropy flux and this brings us to the next result.

Theorem 3.1. (Tadmor 1987*b*, Theorem 5.2) The conservative scheme (3.5) is entropy-stable (respectively, entropy-conservative) if, and for three-point schemes ($p = 1$) only if,

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}_{\nu+\frac{1}{2}} \Big\rangle \le \Delta\psi_{\nu+\frac{1}{2}}, \tag{3.10}$$

and, respectively,

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}_{\nu+\frac{1}{2}} \Big\rangle = \Delta\psi_{\nu+\frac{1}{2}}. \tag{3.11}$$

4. The scalar problem

We discuss the entropy stability of *scalar* schemes of the form (see (3.5))

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[g_{\nu+\frac{1}{2}} - g_{\nu-\frac{1}{2}}\Big], \quad u_\nu(t) \equiv u(v_\nu(t)). \tag{4.1}$$

For a more convenient formulation, let us define for $\Delta v_{\nu+\frac{1}{2}} \neq 0$

$$Q_{\nu+\frac{1}{2}} = \frac{f(u_\nu) + f(u_{\nu+1}) - 2g_{\nu+\frac{1}{2}}}{\Delta v_{\nu+\frac{1}{2}}}, \quad \Delta v_{\nu+\frac{1}{2}} := v_{\nu+1} - v_\nu. \tag{4.2}$$

Our scheme recasts into the equivalent *viscosity form*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = &-\frac{1}{2\Delta x_\nu}\Big[f(u_{\nu+1}) - f(u_{\nu-1})\Big]\\ &+\frac{1}{2\Delta x_\nu}\Big[Q_{\nu+\frac12}\Delta v_{\nu+\frac12} - Q_{\nu-\frac12}\Delta v_{\nu-\frac12}\Big],\end{aligned} \tag{4.3}$$

which reveals the role of $Q_{\nu+\frac12}$ as the numerical viscosity coefficient (*e.g.*, Tadmor (1984*b*)).

According to (3.11), scalar entropy-conservative schemes are uniquely determined by the numerical flux $g_{\nu+\frac12} = g^*_{\nu+\frac12}$, that is,

$$\begin{aligned} g^*_{\nu+\frac12} &:= \frac{\Delta\psi_{\nu+\frac12}}{\Delta v_{\nu+\frac12}} \equiv \int_{\xi=-\frac12}^{\frac12} g\Big(v_{\nu+\frac12}(\xi)\Big)\,\mathrm{d}\xi,\\ v_{\nu+\frac12}(\xi) &:= \frac12(v_\nu + v_{\nu+1}) + \xi\Delta v_{\nu+\frac12}.\end{aligned} \tag{4.4}$$

Noting that

$$g^*_{\nu+\frac12} = \int_{\xi=-\frac12}^{\frac12} \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi)\cdot g\Big(v_{\nu+\frac12}(\xi)\Big)\,\mathrm{d}\xi, \tag{4.5}$$

we find upon integration by parts that entropy-conservative schemes admit the viscosity form (4.3), with a viscosity coefficient $Q_{\nu+\frac12} = Q^*_{\nu+\frac12}$ given by[3]

$$Q^*_{\nu+\frac12} = \int_{\xi=-\frac12}^{\frac12} 2\xi g'\Big(v_{\nu+\frac12}(\xi)\Big)\,\mathrm{d}\xi. \tag{4.6}$$

The entropy-conservative scheme then takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) &= -\frac{1}{\Delta x_\nu}\Big[g^*_{\nu+\frac12} - g^*_{\nu-\frac12}\Big]\\ &= -\frac{1}{2\Delta x_\nu}\big[f(u_{\nu+1}) - f(u_{\nu-1})\big]\\ &\quad+\frac{1}{2\Delta x_\nu}\Big[Q^*_{\nu+\frac12}\Delta v_{\nu+\frac12} - Q^*_{\nu-\frac12}\Delta v_{\nu-\frac12}\Big].\end{aligned} \tag{4.7}$$

The entropy stability portion of Theorem 3.1 can now be restated in the following form.

Corollary 4.1. (Tadmor 1987*b*, Theorem 5.1) The conservative scheme (4.7) and (4.3) is entropy-stable, if – and for three-point schemes ($p = 1$) only if – it contains more viscosity than the entropy-conservative one (4.6),

[3] We use primes to indicate differentiation with respect to primary dependent variables, *e.g.*, $\mathbf{g}' = \mathbf{g}_\mathbf{v}(\mathbf{v}), \mathbf{f}'' = \mathbf{f}_{\mathbf{uu}}(\mathbf{u})$, *etc.*

that is,

$$Q^*_{\nu+\frac{1}{2}} \leq Q_{\nu+\frac{1}{2}}. \tag{4.8}$$

The rest of this section is devoted to examples demonstrating applications of the last corollary.

Example 4.2. (Entropy-conservative schemes) We begin with two examples of entropy-conservative schemes, interesting in their own right, which played a significant role in studying zero dispersion phenomena: see, *e.g.*, Lax (1986) and collaborators.

We consider the inviscid Burgers' equation, $u_t + (\frac{1}{2}u^2)_x = 0$, and we seek a semi-discrete scheme that conserves the logarithmic entropy $U(u) = -\ln u$. The entropy flux in this case is $F(u) = -u$. Using the entropy variable $v(u) = -1/u$, we compute the entropy flux potential

$$\psi(v) = vf(u(v)) - F(u(v)) = -\frac{1}{2v},$$

which in turn yields the entropy-conservative flux

$$g^*_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1} - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{1}{2}\frac{1}{v_\nu v_{\nu+1}} = \frac{1}{2}u_\nu u_{\nu+1}.$$

This yields the entropy-conservative centred schemes

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = u_\nu(t)\frac{u_{\nu+1}(t) - u_{\nu-1}(t)}{2\Delta x_\nu},$$

studied in Goodman and Lax (1988), Hou and Lax (1991) and Levermore and Liu (1996), among others.

Next, we consider $u_t + (\mathrm{e}^u)_x = 0$ and we seek the semi-discrete scheme that conserves the exponential entropy, $U(u) = \mathrm{e}^u$. The entropy flux is $F(u) = \frac{1}{2}\mathrm{e}^{2u}$. Using the corresponding entropy variable $v(u) = \mathrm{e}^u$, we compute the entropy flux potential

$$\psi(v) = vf(u(v)) - F(u(v)) = \frac{1}{2}v^2,$$

which in turn yields the entropy-conservative flux

$$g^*_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{1}{2}(v_\nu + v_{\nu+1}) = \frac{1}{2}\big[\mathrm{e}^{u_\nu} + \mathrm{e}^{u_{\nu+1}}\big].$$

This yields the entropy-conservative centred schemes

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = \frac{\mathrm{e}^{u_{\nu+1}(t)} - \mathrm{e}^{u_{\nu-1}(t)}}{2\Delta x_\nu}$$

associated with Toda flow: consult Lax *et al.* (1993), Levermore and Liu (1996), Deift and McLaughlin (1998), and the references therein.

We continue with a series of entropy-stable examples.

Example 4.3. (Engquist and Osher 1980) Using the estimate

$$Q^*_{\nu+\frac{1}{2}} \le \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left|g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\right| \mathrm{d}\xi = \int \left|f'\Big(u\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\Big)\right| \left|\frac{\mathrm{d}u(v_{\nu+\frac{1}{2}}(\xi))}{\Delta v_{\nu+\frac{1}{2}}}\right|$$
$$= \frac{1}{\Delta v_{\nu+\frac{1}{2}}}\left[\int_{u_\nu}^{u_{\nu+1}} |f'(u)|\,\mathrm{d}u\right] =: Q^{EO}_{\nu+\frac{1}{2}}, \tag{4.9}$$

we obtain an upper bound $Q^{EO}_{\nu+\frac{1}{2}}$, which is the viscosity coefficient associated with the entropy-stable Engquist–Osher (EO) scheme (Engquist and Osher 1980).

The quantity inside the brackets on the right-hand side of (4.9) is independent of different choices for entropy variables. Consequently, the entropy stability of the EO scheme is uniform with respect to all admissible entropy pairs, (U, F). This raises the question of the minimal amount of viscosity required to maintain such uniformity.

Example 4.4. (Godunov 1959) We rewrite the second term on the right-hand side of the schemes (4.3) as

$$\frac{1}{2\Delta x_\nu}\left[\left(Q_{\nu+\frac{1}{2}}\frac{\Delta v_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}\right)\Delta u_{\nu+\frac{1}{2}} - \left(Q_{\nu-\frac{1}{2}}\frac{\Delta v_{\nu-\frac{1}{2}}}{\Delta u_{\nu-\frac{1}{2}}}\right)\Delta u_{\nu-\frac{1}{2}}\right],$$

thus normalizing their viscous part by using the conservative variables as our fixed scale. Since in the scalar case *all* convex functions, $U(u)$, are admissible entropy functions, it follows that, for an entropy stability which is uniform with respect to every such U, we need to *maximize* the corresponding entropy viscous factors $Q^*_{\nu+\frac{1}{2}}\big(\Delta v_{\nu+\frac{1}{2}}/\Delta u_{\nu+\frac{1}{2}}\big)$,

$$\sup_v\left[\frac{f(u_\nu)+f(u_{\nu+1})-2g^*_{\nu+\frac{1}{2}}}{\Delta u_{\nu+\frac{1}{2}}}\right], \quad g^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f\big(u\big(v_{\nu+\frac{1}{2}}(\xi)\big)\big)\,\mathrm{d}\xi,$$

where the supremum is taken over *all increasing* $v = v(u)$. This yields Godunov's viscosity coefficient (Osher 1985, Tadmor 1984*b*)

$$Q^G_{\nu+\frac{1}{2}} = \max_{\min(u_\nu,u_{\nu+1})\le u\le\max(u_\nu,u_{\nu+1})}\left[\frac{f(u_\nu)+f(u_{\nu+1})-2f(u)}{\Delta u_{\nu+\frac{1}{2}}}\right]. \tag{4.10}$$

Thus, the scalar schemes which are uniformly entropy-stable with respect to all convex entropies are precisely those that contain at least as much numerical viscosity as the Godunov scheme does. These so-called E-schemes were first identified in Osher (1984); see also Tadmor (1984*b*).

The E-schemes are only first-order accurate: consult, *e.g.*, Lemma 4.5 below. Corollary 4.1 enables us to verify the entropy stability of second-order accurate schemes as well. To this end we recall from (4.6) that

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \quad v_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}(v_\nu + v_{\nu+1}) + \xi\Delta v_{\nu+\frac{1}{2}}.$$

Integration by parts yields

$$\begin{aligned} Q^*_{\nu+\frac{1}{2}} &= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 - \frac{1}{4}\right) g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \\ &= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi} g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \end{aligned} \tag{4.11}$$

and hence the entropy-conservative viscosity coefficient $Q^*_{\nu+\frac{1}{2}}$ takes the form

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) g''\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \cdot \Delta v_{\nu+\frac{1}{2}}. \tag{4.12}$$

Thus, the viscosity coefficients of the entropy-conservative schemes are in fact of order $\mathcal{O}(|\Delta v_{\nu+\frac{1}{2}}|)$, and this implies their second-order accuracy in view of the following lemma.

Lemma 4.5. Consider the conservative schemes (4.3) with viscosity coefficient, $Q_{\nu+\frac{1}{2}}$, such that $\big(Q_{\nu+\frac{1}{2}}/\Delta v_{\nu+\frac{1}{2}}\big)$ is Lipschitz-continuous. Then these schemes are second-order accurate, in the sense that their local truncation error is of the order

$$\mathcal{O}\Big[|x_{\nu+1} - x_\nu|^2 + |x_\nu - x_{\nu-1}|^2 + |x_{\nu+1} - 2x_\nu + x_{\nu-1}|\Big].$$

Verification of this lemma is straightforward and therefore omitted.

Example 4.6. (Second-order accurate schemes) Using the simple upper bound

$$Q^*_{\nu+\frac{1}{2}} \le \frac{1}{6} \max_{\min(v_\nu, v_{\nu+1}) \le v \le \max(v_\nu, v_{\nu+1})} |g''(v)| \cdot \big|\Delta v_{\nu+\frac{1}{2}}\big|, \tag{4.13}$$

we obtain a viscosity coefficient on the right of (4.13) which, according to Corollary 4.1 and Lemma 4.5, maintains both entropy stability and second-order accuracy. Viscosity terms similar to this were previously derived in a number of special cases, dealing with the entropy stability question of second-order schemes, such as (generalized) van Leer's MUSCL scheme (van Leer 1977, Osher 1985, Lions and Souganidis 1985, Yang 1996*a*) as well as other high-resolution schemes (Majda and Osher 1978, Majda and Osher 1979, Harten 1983*a*, Harten and Hyman 1983, Nessyahu and Tadmor 1990). We remark that the careful calculations required in those derivations are due to the delicate balance of the cubic order of entropy loss, which should match the third-order dissipation in this case.

An instructive example of using the above arguments of entropy stability is provided in the genuinely nonlinear case, where $f(u)$ is, say, convex. A quadratic entropy stability is sufficient in this case, to single out the unique physically relevant solution (Szepessy 1989, Chen 2000). In particular, the choice of the quadratic entropy function $U(u) = \frac{1}{2}u^2$ leads to entropy variables that coincide with the conservative ones, $g(v) = f(u)$. The last three examples of this section deal with this important special case.

Example 4.7. (Lax and Wendroff 1960) By convexity, the entropy-conservative viscosity coefficient in (4.12),

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) f''\Big(u_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \cdot \Delta u_{\nu+\frac{1}{2}}$$

is negative whenever $\Delta u_{\nu+\frac{1}{2}}$ is negative, and hence numerical viscosity is required only in the case of rarefactions where $\Delta u_{\nu+\frac{1}{2}} > 0$. To see how much viscosity is required in this case, we use the fact that the integrand on the right of Q^* is positive, leading to the upper bound

$$Q^*_{\nu+\frac{1}{2}} \leq \frac{1}{4}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f''\Big(u_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \cdot \Delta u_{\nu+\frac{1}{2}} = \frac{1}{4}\big[a(u_{\nu+1}) - a(u_\nu)\big]^+. \quad (4.14)$$

The resulting viscosity coefficient on the right is the second-order accurate viscosity originally proposed by Lax and Wendroff (1960),

$$Q^{LW}_{\nu+\frac{1}{2}} = \frac{1}{4}\big[a(u_{\nu+1}) - a(u_\nu)\big]^+, \quad a(u) = f'(u). \quad (4.15)$$

Example 4.8. (Centred schemes) According to (2.9) with $g(v) = f(u)$, the entropy flux potential is given by the primitive of $f(\cdot)$, and by (3.10), entropy stability is guaranteed if

$$\Delta u_{\nu+\frac{1}{2}} \cdot f_{\nu+\frac{1}{2}} \leq \int_{u_\nu}^{u_{\nu+1}} f(u)\,\mathrm{d}u.$$

In the rarefaction case, $\Delta u_{\nu+\frac{1}{2}} > 0$, the integral on the right approximated from below by the midpoint rule; in the case of a shock, $\Delta u_{\nu+\frac{1}{2}} < 0$, signs are reversed and we can instead use the trapezoidal rule. Thus we derive a second-order accurate entropy stable scheme (4.1), whose simple numerical flux is given by the centred numerical flux

$$g_{\nu+\frac{1}{2}} = f_{\nu+\frac{1}{2}} = \begin{cases} f\Big(\dfrac{u_\nu + u_{\nu+1}}{2}\Big), & \Delta u_{\nu+\frac{1}{2}} > 0 \\[2ex] \dfrac{f(u_\nu) + f(u_{\nu+1})}{2}, & \Delta u_{\nu-\frac{1}{2}} \le 0. \end{cases} \tag{4.16}$$

We conclude this section with the following example.

Example 4.9. A well-known 'trick' for deriving a quadratic entropy-conservative approximation in the particular case of the inviscid Burgers' equation

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left[\frac{1}{2}u^2\right] = 0, \tag{4.17}$$

is based on centred differencing of its equivalent skew-adjoint form (Tadmor 1984*a*)

$$\frac{\partial}{\partial t}u + \frac{1}{3}\frac{\partial}{\partial x}[u^2] + \frac{1}{3}u\frac{\partial}{\partial x}[u] = 0,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = -\frac{1}{3}\frac{1}{2\Delta x_\nu}\big[u^2_{\nu+1} - u^2_{\nu-1}\big] - \frac{1}{3}u_\nu\frac{1}{2\Delta x_\nu}\big[u_{\nu+1} - u_{\nu-1}\big].$$

In fact, there is more than just a 'trick' here: the resulting scheme is simply a special case of our entropy-conservative recipe (4.6)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = &-\frac{1}{2\Delta x_\nu}\left[\frac{1}{2}u^2_{\nu+1} - \frac{1}{2}u^2_{\nu-1}\right] \\ &+ \frac{1}{2\Delta x_\nu}\left[\frac{1}{6}\Big(\Delta u_{\nu+\frac{1}{2}}\Big)^2 - \frac{1}{6}\Big(\Delta u_{\nu-\frac{1}{2}}\Big)^2\right].\end{aligned} \tag{4.18}$$

If we exclude negative viscosity, however, then according to (4.12) the least viscous entropy-stable approximation of Burgers' equation (4.17) is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = &-\frac{1}{2\Delta x_\nu}\left[\frac{1}{2}u^2_{\nu+1} - \frac{1}{2}u^2_{\nu-1}\right] \\ &+ \frac{1}{2\Delta x_\nu}\left[\frac{1}{6}\Big(\Delta u_{\nu+\frac{1}{2}}\Big)^+\Delta u_{\nu+\frac{1}{2}} - \frac{1}{6}\Big(\Delta u_{\nu-\frac{1}{2}}\Big)^+\Delta u_{\nu-\frac{1}{2}}\right].\end{aligned} \tag{4.19}$$

5. Systems of conservation laws

We study the entropy stability of the semi-discrete schemes that are consistent with the *system* of conservation laws (2.5). The schemes assume the following viscosity form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] + \frac{1}{2\Delta x_\nu}\Big[Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big]. \tag{5.1}$$

Difference schemes that admit the viscosity form (5.1) are precisely the so-called essentially three-point schemes (Harten (1983*a*), Tadmor (1987*b*, Lemma 5.1)), namely, difference schemes whose numerical flux, $\mathbf{f}(\cdot,\cdot,\cdots)$ satisfies the restricted consistency relation

$$\mathbf{f}(\mathbf{u}_{\nu-p+1},\ldots,\mathbf{u}_\nu = \mathbf{u}_{\nu+1} = \mathbf{u},\ldots,\mathbf{u}_{\nu+p}) = \mathbf{f}(\mathbf{u}).$$

This is the case of (5.1), with

$$\mathbf{f}(\ldots,\mathbf{u}_\nu,\mathbf{u}_{\nu+1},\ldots) = \frac{1}{2}\big[\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})\big] - \frac{1}{2}Q_{\nu+\frac{1}{2}}(\mathbf{v}_{\nu+1} - \mathbf{v}_\nu),$$
$$\mathbf{v}_\nu = \mathbf{v}(\mathbf{u}_\nu).$$

A couple of remarks are in order.

(1) The class of essentially three-point schemes includes classical schemes based on three-point stencils ($p = 1$), as well as most modern high-resolution schemes (van Leer 1977, Harten 1983*a*); consult Godlewski and Raviart (1996), Kröner (1997), LeVeque (1992, 2002) and the references therein.

(2) The use of essentially three-point stencils in this section is linked to the specific second-order entropy-conservative schemes discussed below. Extensions to higher orders and larger stencils were carried out by LeFloch and Rohde (2000, Section 4); consult Section 9 below.

To extend our scalar entropy stability analysis to systems of conservation laws we proceed as before, by comparison with certain entropy-conservative schemes. Unlike the scalar problem, however, we now have more than one way to meet the entropy conservation requirement (3.11). The various ways differ in their choice of the path of integration in phase space. In this section, we restrict our attention to the simplest choice along the *straight path* $\mathbf{v}_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}$. The corresponding entropy-conservative flux is given by

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi,$$
$$\mathbf{v}_{\nu+\frac{1}{2}}(\xi) := \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}. \tag{5.2}$$

Indeed, the entropy conservation requirement (3.1) is fulfilled in this case, since, in view of (2.9),

$$\Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \Big\rangle = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big\rangle \,\mathrm{d}\xi$$
$$= \int_{\mathbf{v}_\nu}^{\mathbf{v}_{\nu+1}} \langle \,\mathrm{d}\mathbf{v}, \mathbf{g}(\mathbf{v})\rangle = \Delta \psi_{\nu+\frac{1}{2}}.$$

The entropy-conservative flux (5.2) was introduced in Tadmor (1986*b*, 1987*b*). As before (see (4.5), (4.6)), we integrate by parts to find

$$\begin{aligned}\mathbf{g}^*_{\nu+\frac{1}{2}} &= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi)\mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \\ &= \xi \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big|_{\xi=-\frac{1}{2}}^{\frac{1}{2}} - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi \mathbf{g}_{\mathbf{v}}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\frac{\mathrm{d}\mathbf{v}_{\nu+\frac{1}{2}}(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi \\ &= \frac{1}{2}\big[\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})\big] - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \Delta \mathbf{v}_{\nu+\frac{1}{2}}. \end{aligned} \tag{5.3}$$

Thus, the entropy-conservative scheme (5.2) admits the equivalent viscosity form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = &-\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] \\ &+ \frac{1}{2\Delta x_\nu}\Big[Q^*_{\nu+\frac{1}{2}}\Delta \mathbf{v}_{\nu+\frac{1}{2}} - Q^*_{\nu-\frac{1}{2}}\Delta \mathbf{v}_{\nu-\frac{1}{2}}\Big],\end{aligned} \tag{5.4}$$

with a numerical viscosity matrix coefficient, $Q^*_{\nu+\frac{1}{2}}$, given by

$$Q^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \quad B(\mathbf{v}) = \mathbf{g}_{\mathbf{v}}(\mathbf{v}). \tag{5.5}$$

The entropy stability portion of Theorem 3.1 can now be conveniently interpreted as follows.

Corollary 5.1. The conservative scheme (5.1) is entropy-stable if – and for three-point schemes ($p = 1$) only if – it contains more viscosity than the entropy-conservative one (5.4), (5.5), that is,

$$\Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta \mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \le \Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}}\Delta \mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle. \tag{5.6}$$

Indeed, we can provide a precise measure for the amount of entropy dissipation in terms of the *dissipation matrix* $D_{\nu+\frac{1}{2}} \equiv D_{\nu+\frac{1}{2}}(\mathbf{v}(t)) := Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}}$

(Tadmor 1987*b*, Theorem 5.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] \tag{5.7}$$
$$= -\frac{1}{4\Delta x_\nu}\Big[\Big\langle \Delta\mathbf{v}_{\nu-\frac{1}{2}}, D_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big\rangle + \frac{1}{4}\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big]$$

Here, $F_{\nu+\frac{1}{2}}$ stands for the entropy flux (see (3.8))

$$F_{\nu+\frac{1}{2}} = \frac{1}{2}\Big\langle \mathbf{v}_\nu + \mathbf{v}_{\nu+1}, \mathbf{g}^*_{\nu+\frac{1}{2}}\Big\rangle - \frac{1}{2}\big[\psi(\mathbf{v}_\nu) + \psi(\mathbf{v}_{\nu+1})\big]$$
$$- \frac{1}{4\Delta x_\nu}\Big\langle \mathbf{v}_\nu + \mathbf{v}_{\nu+1}, D_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle. \tag{5.8}$$

The entropy-conservative flux (5.2), and likewise its corresponding viscosity coefficient in (5.5), cannot be evaluated in a closed form. However, Corollary 5.1 enables us to verify entropy stability by *comparison*, $Q^*_{\nu+\frac{1}{2}} \le \mathrm{Re}\, Q_{\nu+\frac{1}{2}}$, with the usual ordering between symmetric matrices. We note in passing that $Q^*_{\nu+\frac{1}{2}}$ is symmetric (since $B(\cdot)$ is) and that, in the generic case, the viscosity coefficient $Q_{\nu+\frac{1}{2}}$ is also symmetric. The following examples demonstrate this point.

Example 5.2. (Rusanov 1961, Lax 1954) We seek a scalar viscosity coefficient, $p_{\nu+\frac{1}{2}}I_N$, which guarantees the entropy stability of the scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - f(\mathbf{u}_{\nu-1})\big]$$
$$+ \frac{1}{2\Delta x_\nu}\Big[p_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - p_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\Big]. \tag{5.9}$$

Using (2.6) we have

$$\Delta\mathbf{u}_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \cdot \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \tag{5.10}$$

and hence the viscous part of the scheme (5.9) can be interpreted in terms of the entropy variables (rather than the conservative ones), as

$$p_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} = Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}, \tag{5.11}$$

where

$$Q_{\nu+\frac{1}{2}} = p_{\nu+\frac{1}{2}}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H(\xi)\,\mathrm{d}\xi, \quad H(\xi) \equiv H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big). \tag{5.12}$$

Recalling (5.5), we conclude that entropy stability is guaranteed by Corollary 5.1, provided $p_{\nu+\frac{1}{2}}$ is chosen so that the inequality

$$2\xi B(\xi) \leq p_{\nu+\frac{1}{2}} H(\xi), \quad B(\xi) \equiv B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big), \quad -\frac{1}{2} \leq \xi \leq \frac{1}{2},$$

holds. To this end, multiply both sides by $H^{-\frac{1}{2}}(\xi)$; by congruence, we end up with the equivalent inequality[4]

$$2\xi \sup_{\lambda} \lambda\big[H^{-\frac{1}{2}}(\xi)B(\xi)H^{-\frac{1}{2}}(\xi)\big] \leq p_{\nu+\frac{1}{2}} I_N. \tag{5.13}$$

We recall that $B = \mathbf{g_v} = \mathbf{f_u}\mathbf{u_v} = AH$, that is,

$$B(\xi) = A(\xi)H(\xi), \quad A(\xi) \equiv A\Big(\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big). \tag{5.14}$$

Hence (5.13) holds and entropy stability follows, for any scalar $p_{\nu+\frac{1}{2}}$ satisfying

$$\begin{aligned} p_{\nu+\frac{1}{2}} &\geq \max_{\lambda, |\xi| \leq \frac{1}{2}} \Big|2\xi\lambda\Big[H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi)\Big]\Big| \\ &= \max_{\lambda, |\xi| \leq \frac{1}{2}} \Big|\lambda\Big[A\Big(\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big)\Big]\Big|. \end{aligned} \tag{5.15}$$

The cell-dependent viscosity factor on the right corresponds to the Rusanov scheme (Rusanov 1961; see also Richtmyer and Morton (1967, Section 2)), while a uniform viscosity factor, satisfying

$$p_{\nu+\frac{1}{2}} \equiv p \geq \max_{\lambda, \mathbf{u}} |\lambda[A(\mathbf{u})]|,$$

corresponds to a Lax–Friedrichs viscosity (Friedrichs 1954, Lax 1954). Both schemes are entropy-stable with respect to any entropy pair associated with equation (2.1).

The last example was restricted to first-order accurate schemes. Yet Corollary 5.1 can be used to maintain both entropy stability and second-order accuracy, as was done in the scalar case. To this end, we proceed as follows. Using (5.14) we can rewrite the quantity on the left of (5.6) as

$$\begin{aligned} &\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}} \Delta\mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \\ &\quad = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi \Big\langle H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}, H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi) \cdot H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \,\mathrm{d}\xi. \end{aligned} \tag{5.16}$$

Let $\{a_k(\xi), \mathbf{r}_k(\xi)\}_{k=1}^N$ be the eigenpairs of $A(\xi)$, that is,

$$a_k(\xi) \equiv a^{(k)}_{\nu+\frac{1}{2}}\Big(\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big), \qquad \mathbf{r}_k(\xi) \equiv \mathbf{r}^{(k)}_{\nu+\frac{1}{2}}\Big(\mathbf{u}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big).$$

[4] Here and below, $\lambda_k[\cdot]$ denotes the kth eigenvalue of a matrix.

Since $H^{-\frac{1}{2}}(\xi)\mathbf{r}_k(\xi)$ are the eigenvectors of the matrix $H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi)$, and since, by (5.14),

$$H^{-\frac{1}{2}}(\xi)A(\xi)H^{\frac{1}{2}}(\xi) \equiv H^{-\frac{1}{2}}(\xi)B(\xi)H^{-\frac{1}{2}}(\xi), \quad B(\xi) \equiv B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big), \tag{5.17}$$

is a symmetric matrix, it follows after normalization that $\{H^{-\frac{1}{2}}(\xi)\mathbf{r}_k(\xi)\}$ form an orthonormal system, that is,

$$\Big\langle H^{-\frac{1}{2}}(\xi)\mathbf{r}_k(\xi), H^{-\frac{1}{2}}(\xi)\mathbf{r}_j(\xi)\Big\rangle = \delta_{jk}. \tag{5.18}$$

We expand $H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}$ and substitute the expansion into the right-hand side of (5.16) to find

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle = \sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi a_k(\xi)\cdot\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2\,\mathrm{d}\xi. \tag{5.19}$$

Finally, we integrate by parts along the lines of (4.11), arriving at

$$\begin{aligned}
&\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \\
&\quad = \sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{4}-\xi^2\Big)\frac{\mathrm{d}}{\mathrm{d}\xi}a_k(\xi)\cdot\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2\,\mathrm{d}\xi \\
&\qquad + \sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{4}-\xi^2\Big)a_k(\xi)\cdot\frac{\mathrm{d}}{\mathrm{d}\xi}\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2\,\mathrm{d}\xi.
\end{aligned} \tag{5.20}$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}\xi}a_k(\xi)\cdot\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2 = \Big\langle \nabla_{\mathbf{v}}a_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\cdot\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2 = 2\cdot\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\cdot\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \nabla_{\mathbf{v}}\mathbf{r}_k(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle,$$

and since both terms are of order $\mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^3\big)$, it follows that the quantity on the right of (5.20) does not exceed

$$\Big|\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big| \le C_{\nu+\frac{1}{2}}\cdot\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^3. \tag{5.21}$$

Thus, the entropy-conservative schemes (5.2) dissipate entropy at a cubic rate and are therefore second-order accurate: consult Lemma 4.5. Comparison of this in light of Corollary 5.1 yields the following entropy stability criterion which respects second-order accuracy.

Theorem 5.3. The conservative scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] + \frac{1}{2\Delta x_\nu}\Big[Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big] \tag{5.22}$$

is entropy-stable if the eigenvalues of (the symmetric part of) its viscosity coefficient matrix, $\operatorname{Re} Q_{\nu+\frac{1}{2}}$, satisfy

$$\min_\lambda \lambda\Big[\operatorname{Re} Q_{\nu+\frac{1}{2}}\Big] \geq C_{\nu+\frac{1}{2}} \cdot \big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|. \tag{5.23}$$

Next, we would like to convert this entropy stability criterion to difference schemes which are written in the standard form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] + \frac{1}{2\Delta x_\nu}\Big[P_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} - P_{\nu-\frac{1}{2}}\Delta\mathbf{u}_{\nu-\frac{1}{2}}\Big]. \tag{5.24}$$

Thus, the viscous part is expressed entirely in terms of the conservative variables, $\mathbf{u}$, instead of the entropy variables, $\mathbf{v}$, used in (5.22). From the corresponding differential set-up, (2.10), we already know that the admissible viscosity matrices for this formulation are those Ps for which $PU_{\mathbf{uu}}^{-1}$ are symmetric positive definite, (2.11), $PH - HP^\top = 0$; consequently, a discrete analogue should hold, at least to leading order, that is,

$$\big\|P_{\nu+\frac{1}{2}}H_{\nu+\frac{1}{2}} - H_{\nu+\frac{1}{2}}P^\top_{\nu+\frac{1}{2}}\big\| \leq \delta_{\nu+\frac{1}{2}}\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big|. \tag{5.25}$$

Here, $H_{\nu+\frac{1}{2}}$ can be any first-order symmetric approximation to the inverse Hessian, $H = U_{\mathbf{uu}}^{-1}$,

$$H_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi + \mathcal{O}\Big(\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big|\Big), \tag{5.26}$$
$$0 < \frac{1}{K}\cdot I_N \leq H_{\nu+\frac{1}{2}} \leq K \cdot I_N.$$

Theorem 5.4. Consider the conservative difference scheme (5.24) with numerical viscosity coefficient $P_{\nu+\frac{1}{2}}$, which is essentially H-symmetric (5.25), (5.26). The scheme is entropy-stable if the eigenvalues of its viscosity coefficient matrix $\lambda(P_{\nu+\frac{1}{2}})$ satisfy

$$\min_\lambda \lambda\Big[P_{\nu+\frac{1}{2}}\Big] \geq \gamma_{\nu+\frac{1}{2}}\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big|. \tag{5.27}$$

Here $\gamma_{\nu+\frac{1}{2}}$ is a suitably large constant depending on (5.21), (5.25), (5.26) for which

$$\gamma_{\nu+\frac{1}{2}} \geq K\Big(C_{\nu+\frac{1}{2}} + \delta_{\nu+\frac{1}{2}}\Big). \tag{5.28}$$

Proof. Using (5.10), the scheme (5.24) meets the desired form in (5.22) with

$$Q_{\nu+\frac{1}{2}} = P_{\nu+\frac{1}{2}}H, \quad H \equiv H_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi.$$

Next, we invoke the identity

$$H^{-\frac{1}{2}}P_{\nu+\frac{1}{2}}H^{\frac{1}{2}} = H^{-\frac{1}{2}}\Big(\operatorname{Re}Q_{\nu+\frac{1}{2}}\Big)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}\left(\frac{P_{\nu+\frac{1}{2}}H - HP^{\top}_{\nu+\frac{1}{2}}}{2}\right)H^{-\frac{1}{2}}.$$

According to (5.27), the eigenvalues of the matrix on the left are bounded from below by $\gamma_{\nu+\frac{1}{2}} \cdot |\Delta\mathbf{u}_{\nu+\frac{1}{2}}|$. Hence, by (5.25), (5.26), the same is true for the eigenvalues of the first matrix on the right; more precisely, we have

$$\lambda\Big[H^{-\frac{1}{2}}\Big(\operatorname{Re}Q_{\nu+\frac{1}{2}}\Big)H^{-\frac{1}{2}}\Big] \geq \Big(\gamma_{\nu+\frac{1}{2}} - K\delta_{\nu+\frac{1}{2}}\Big)\cdot\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big|.$$

Multiplying on both sides by $H^{\frac{1}{2}}$ we find, on account of (5.28),

$$\lambda\Big[\operatorname{Re}Q_{\nu+\frac{1}{2}}\Big] \geq \left(\frac{1}{K}\gamma_{\nu+\frac{1}{2}} - \delta_{\nu+\frac{1}{2}}\right)\cdot\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big| \geq C_{\nu+\frac{1}{2}}\cdot\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|,$$

and entropy stability follows from Theorem 5.3. □

Equipped with Theorem 5.4 we turn to the following example.

Example 5.5. (Second-order accurate scalar numerical viscosity) We re-examine Example 5.2, considering the case of scalar viscosity in (5.9), where $p_{\nu+\frac{1}{2}} = p_{\nu+\frac{1}{2}}I_N$. By Theorem 5.4, any scalar satisfying

$$p_{\nu+\frac{1}{2}} \geq KC_{\nu+\frac{1}{2}}\cdot\big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big| \tag{5.29}$$

will guarantee entropy stability as well as maintain second-order accuracy.

Example 5.6. (Roe-type schemes) We consider the class of schemes based on Roe's decomposition (Roe 1981). To this end, we introduce a Lipschitz-continuous averaged Jacobian, the so-called Roe matrix, $\overline{A}_{\nu+\frac{1}{2}}$, satisfying

$$\Delta\mathbf{f}_{\nu+\frac{1}{2}} \equiv \overline{A}_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}}, \qquad \Delta\mathbf{f}_{\nu+\frac{1}{2}} := \mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu}), \tag{5.30}$$

and having a complete real eigensystem. Roe (1981) constructed such a matrix for the Euler equations (2.15); Harten and Lax (1981) have shown its existence in the general case, namely

$$\overline{A}_{\nu+\frac{1}{2}} = B_{\nu+\frac{1}{2}} \cdot H^{-1}_{\nu+\frac{1}{2}}, \tag{5.31}$$

where $B_{\nu+\frac{1}{2}}$ and $H_{\nu+\frac{1}{2}}$ are defined by the cell averages

$$B_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\, \mathrm{d}\xi, \quad H_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} H\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\, \mathrm{d}\xi. \tag{5.32}$$

Given a Roe matrix, the viscosity coefficient in (5.24), $P_{\nu+\frac{1}{2}}$, is then set to be

$$P_{\nu+\frac{1}{2}} = p\Big(\overline{A}_{\nu+\frac{1}{2}}\Big). \tag{5.33}$$

Here $p(\cdot)$ is an appropriate viscosity function which is computed according to the spectral decomposition of $\overline{A}_{\nu+\frac{1}{2}}$, namely,

$$p\Big(\overline{A}_{\nu+\frac{1}{2}}\Big) = R_{\nu+\frac{1}{2}} \cdot \begin{bmatrix} p(\overline{a}_1) & & \\ & \ddots & \\ & & p(\overline{a}_N) \end{bmatrix} \cdot R^{-1}_{\nu+\frac{1}{2}}. \tag{5.34}$$

where $\{(\overline{a})_1^N, R_{\nu+\frac{1}{2}}\}$ is the eigensystem of $\overline{A}$, that is,

$$\overline{A}_{\nu+\frac{1}{2}} = R_{\nu+\frac{1}{2}} \cdot \begin{bmatrix} \overline{a}_1 & & \\ & \ddots & \\ & & \overline{a}_N \end{bmatrix} \cdot R^{-1}_{\nu+\frac{1}{2}}, \quad \overline{a}_k := \lambda_k\Big[\overline{A}_{\nu+\frac{1}{2}}\Big].$$

For a given system, the possibly various choices of a Roe matrices are within $\mathcal{O}(|\Delta\mathbf{u}_{\nu+\frac{1}{2}}|)$ of each other; since the set-up of Theorem 5.4 is invariant under such perturbations we can discuss without restriction the one choice given in (5.31)–(5.32). With this choice of a Roe matrix we have

$$P_{\nu+\frac{1}{2}} = p\Big(\overline{A}_{\nu+\frac{1}{2}}\Big) = H^{\frac{1}{2}}_{\nu+\frac{1}{2}} \cdot p\Big[H^{-\frac{1}{2}}_{\nu+\frac{1}{2}} \cdot \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\, \mathrm{d}\xi \cdot H^{-\frac{1}{2}}_{\nu+\frac{1}{2}}\Big] \cdot H^{-\frac{1}{2}}_{\nu+\frac{1}{2}},$$

and hence, by the symmetry of B, it follows that P is H-symmetric, so that (5.25) holds with $\delta_{\nu+\frac{1}{2}} = 0$. Theorem 5.4 applies and we are led to the following.

Theorem 5.7. The conservative Roe-type scheme (5.24), (5.33), (5.34) is entropy-stable, provided that its viscosity function $p(\cdot)$ satisfies

$$p(\overline{a}_k) \geq KC_{\nu+\frac{1}{2}} \cdot \big|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\big|, \quad \overline{a}_k := \lambda_k\Big[\overline{A}_{\nu+\frac{1}{2}}\Big]. \tag{5.35}$$

There are various ways of choosing a viscosity function $p(\cdot)$ satisfying (5.35), which give rise to either first- or second-order accurate entropy-stable schemes. We start by discussing the pros and cons of some first-order choices.

The original choice of Roe (1981) employs the viscosity function

$$p(\overline{a}_k) = |\overline{a}_k|. \tag{5.36}$$

It has the desirable property that discrete steady shocks are perfectly resolved on the grid. With this choice, the entropy stability requirement (5.35) reads

$$|\overline{a}_k| \geq KC_{\nu+\frac{1}{2}} \cdot \left|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\right|,$$

and it is fulfilled as long as we are away from sonic points. Yet this requirement may be violated in sonic neighbourhoods where $\overline{a}_k \approx 0$; indeed, Roe's scheme is the canonical example of an entropy-unstable scheme, for it admits steady expansion shocks. Theorem 5.7 suggests a simple modification – first proposed by Osher (1985, Theorem 3.3) – in order to avoid such instability.

Example 5.8. (First-order entropy fix of the Roe scheme) The Roe scheme (5.24), (5.33), (5.34), is entropy-stable with a viscosity function

$$p(\overline{a}_k) = \max\left\{|\overline{a}_k|, KC_{\nu+\frac{1}{2}} \cdot \left|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\right|\right\}. \tag{5.37}$$

The slightly more viscous modification of Harten (1983*a*) takes the form

$$p(\overline{a}_k) = \max\left\{|\overline{a}_k|, \varepsilon\right\}, \quad \left|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\right| \ll 1. \tag{5.38}$$

In these cases entropy stability is achieved by adding viscosity near sonic points, regardless of whether they occur in rarefaction or shock waves. This is done at the expense of destroying the sharp steady shock resolution of Roe's original scheme (5.36).

However, we can do better with regard to Roe-type schemes, by sharpening the general sufficient entropy stability condition (5.27) which led us to Theorem 5.7. To this end, we first note that the eigensystem of a Roe matrix in (5.30), $\overline{A}_{\nu+\frac{1}{2}}$, is within $\mathcal{O}(|\Delta\mathbf{u}_{\nu+\frac{1}{2}}|^2)$ from the eigensystem of the exact mid-value Jacobian, say $A(\xi = 0)$, for example,

$$|\overline{\mathbf{r}}_k - \mathbf{r}_k(\xi = 0)| + |\overline{a}_k - a_k(\xi = 0)| \leq \text{Const}\left|\Delta\mathbf{u}_{\nu+\frac{1}{2}}\right|^2. \tag{5.39}$$

By virtue of (5.39) we can obtain rather detailed information about the entropy dissipation rate of the Roe-type schemes (5.24), (5.33), (5.34).

Let $\Delta a_k(\mathbf{u}_\nu)$ denote the jump in the kth eigenvalue

$$\Delta a_k(\mathbf{u}_\nu) = \lambda_k(A(\mathbf{u}_{\nu+1})) - \lambda_k(A(\mathbf{u}_\nu)).$$

In the Appendix we prove the following theorem.

Theorem 5.9. The conservative Roe-type scheme (5.24), (5.33), (5.34) is entropy-stable if its viscosity function, $p(\cdot)$, satisfies

$$p(\overline{a}_k) \geq \frac{1}{6}\left[\Delta a_k(\mathbf{u}_\nu) + \varepsilon_k|\overline{a}_k| + \left(1+\frac{|\overline{a}_k|}{\varepsilon_k}\right)\text{Const}\left|\Delta \mathbf{u}_{\nu+\frac{1}{2}}\right|^2\right]. \tag{5.40}$$

Here $\varepsilon_k > 0$ are arbitrary parameters at our disposal.

Remark. We note that the essential ingredient of Theorem 5.9 is not the Roe averaging property (5.30), but the requirement that the eigensystem of $\overline{A}_{\nu+\frac{1}{2}}$ be within $\mathcal{O}\left(\left|\Delta \mathbf{u}_{\nu+\frac{1}{2}}\right|^2\right)$ of the eigensystem of $A(\xi = 0)$, (5.39). Hence, Theorem 5.9 and its consequences apply to other Roe averages, for example, $\overline{A}_{\nu+\frac{1}{2}} = A[\frac{1}{2}(\mathbf{u}_\nu + \mathbf{u}_{\nu+1})]$ or $\overline{A}_{\nu+\frac{1}{2}} = \frac{1}{2}\left[A(\mathbf{u}_\nu) + A(\mathbf{u}_{\nu+1})\right]$.

We shall apply Theorem 5.9 to hyperbolic systems which contain either *genuinely nonlinear* (GNL) or linearly degenerate waves. Lax (1957) has shown that any two nearby states in such systems, $\mathbf{u}_\nu$ and $\mathbf{u}_{\nu+1}$, can be connected by a certain continuous path in phase space; the jump from $\mathbf{u}_\nu$ to $\mathbf{u}_{\nu+1}$ is resolved into a succession of k-waves, $k = 1, 2, \ldots, N$, each of which is either a k-shock, a k-contact or a k-rarefaction, depending on whether a_k increases, remains constant or decreases, respectively, along the corresponding kth subpath. In this section we answer the entropy stability question by comparison with the entropy-conservative schemes (5.2) which are based upon integration along a simple straight path in phase space. Therefore, these schemes do not resolve the full structure in phase space of the solution path to the Riemann problem described above. Instead, we shall confine ourselves to identify each cell with one dominant k-wave. This fact of one dominant wave per cell is certainly the case with GNL scalar problems and, as observed by Harten (1983*a*), is also valid in actual computations with the gas dynamics system (2.15). We will refine our stability analysis in Section 6 below, in terms of new entropy conservative schemes which do take into account different subpaths in phase space.

Choosing $\varepsilon_k = 6$ in (5.40), then the following entropy-stable modification of the first-order Roe-type scheme is obtained.

Example 5.10. (Modified Roe scheme revisited) The conservative Roe-type scheme (5.24), (5.33), (5.34) is entropy-stable with viscosity function

$$p(\overline{a}_k) = |\overline{a}_k| + \left[\frac{1}{6}\Delta a_k(\mathbf{u}_\nu) + \text{Const}\left|\Delta \mathbf{u}_{\nu+\frac{1}{2}}\right|^2\right]^+. \tag{5.41}$$

This choice of viscosity function was suggested in Harten and Hyman (1983, Appendix A) and numerical simulations were carried out in Kaddouri (1993), for example. To gain a better insight into this choice, we shall distinguish between three different cases.

Case I. $\Delta a_k(\mathbf{u}_\nu) \le -\text{Const}\big|\Delta\mathbf{u}_{\nu+\frac12}\big| + \mathcal{O}\big(\big|\Delta\mathbf{u}_{\nu+\frac12}\big|^2\big)$.
In this case the jump Δa_k is dominated by a k-shock, and with sufficiently small variation, no additional viscosity is required in (5.36), *i.e.*, (5.41) is reduced to $p(\overline{a}_k) = |\overline{a}_k|$. Thus, (5.41) retains the perfect resolution of (sufficiently weak) discrete steady shocks.

Case II. $\Delta a_k(\mathbf{u}_\nu) = \mathcal{O}\big(\big|\Delta\mathbf{u}_{\nu+\frac12}\big|^2\big)$.
In this case the jump Δa_k is essentially due to the k-contact field and/or the balance between the other fields. Here, a minimal amount of viscosity is required near sonic points $p(\overline{a}_k) = |\overline{a}_k| + \text{Const}\big|\Delta\mathbf{u}_{\nu+\frac12}\big|^2$.

Case III. Finally, in all other cases we shall identify the jump $\Delta a_k(\mathbf{u}_\nu)$ as dominated by a k-rarefaction, and as expected, $\mathcal{O}\big(\big|\Delta\mathbf{u}_{\nu+\frac12}\big|\big)$ amount of dissipation is required near sonic points, $p(\overline{a}_k) = |\overline{a}_k| + \text{Const}\big|\Delta\mathbf{u}_{\nu+\frac12}\big|$.

This concludes our discussion of the first-order accurate Roe-type schemes and we turn to the second-order case. Choosing $\varepsilon_k \sim \big|\Delta\mathbf{u}_{\nu+\frac12}\big|$ in (5.40), we find $p(\overline{a}_k) = \Delta a_k(\mathbf{u}_\nu) + \text{Const}\big(|\overline{a}_k|\Delta\mathbf{u}_\nu + |\Delta\mathbf{u}_\nu|^2\big)$. Thus, if we set

$$p(\overline{a}_k) = \text{Const}\big|\Delta\mathbf{u}_{\nu+\frac12}\big|, \tag{5.42}$$

then the resulting Roe-type scheme (5.24), (5.33), (5.34) is second-order accurate by Lemma 4.5, and it is entropy-stable for sufficiently large Const $\ge KC_{\nu+\frac12}$: consult Theorem 5.7.

The examples studied so far are based on *a priori* (positive) bounds for the entropy-conservative viscosity. We close this section with the following example, which shows how to enforce entropy stability *a posteriori* by carefully removing any viscosity production. We start with an essentially three-point scheme in its conservative variables formulation (5.24),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = &-\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] \\ &+ \frac{1}{2\Delta x_\nu}\Big[P_{\nu+\frac12}\Delta\mathbf{u}_{\nu+\frac12} - P_{\nu-\frac12}\Delta\mathbf{u}_{\nu-\frac12}\Big],\end{aligned} \tag{5.43}$$

and we compare it with the corresponding formulation of the conservative scheme (5.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = &-\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] \\ &+ \frac{1}{2\Delta x_\nu}\Big[P^*_{\nu+\frac12}\Delta\mathbf{u}_{\nu+\frac12} - P^*_{\nu-\frac12}\Delta\mathbf{u}_{\nu-\frac12}\Big],\end{aligned}$$

$$\text{where}\quad P^*_{\nu+\frac12} := Q^*_{\nu+\frac12}(H^{-1})^*, \quad (H^{-1})^* := \int_{\xi=-\frac12}^{\frac12} H^{-1}\Big(\mathbf{u}_{\nu+\frac12}(\xi)\Big)\,\mathrm{d}\xi.$$

According to (5.7) which we express in terms of the conservative variables, the entropy production of (5.43) is quantified by

$$e_{\nu+\frac{1}{2}} := \left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, E_{\nu+\frac{1}{2}}\Delta\mathbf{u}_{\nu+\frac{1}{2}} \right\rangle, \quad E_{\nu+\frac{1}{2}} := P^*_{\nu+\frac{1}{2}} - P_{\nu+\frac{1}{2}},$$

and we arrive at the following example.

Example 5.11. (Khalfallah and Lerat 1988) The scheme (5.43) becomes entropy-stable if we add a minimal amount of (scalar) viscosity correction, $p^c_{\nu+\frac{1}{2}}$, replacing $P_{\nu+\frac{1}{2}}$ with

$$P_{\nu+\frac{1}{2}} \longrightarrow P_{\nu+\frac{1}{2}} + p^c_{\nu+\frac{1}{2}} I_{N\times N}, \quad p^c := \frac{(e_{\nu+\frac{1}{2}})^+}{\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{u}_{\nu+\frac{1}{2}} \right\rangle}. \tag{5.44}$$

We note that the quantity on the right is well defined since the denominator does not vanish $\langle (H^{-1})^*\mathbf{u}, \mathbf{u}\rangle > 0$. The correction preserves second-order accuracy, and Lerat and his co-workers (Khalfallah and Lerat 1988) report on successful applications of such entropy correction in numerical simulations of fluid dynamics problems. Let us point out two limitations to the present approach: (i) we need to compute the entropy-conservative term $P^*\mathbf{u} = Q^*\mathbf{v}$, which might not be readily available, and (ii), as before, the entropy correction does not distinguish between different waves within the same cell. Both points are addressed in the context of the new entropy-conservative schemes introduced in the next section: consult (6.12) below, for example.

6. Entropy-conservative schemes revisited

Our study of entropy stability is based on comparison with entropy-conservative schemes. In the scalar case, entropy-conservative schemes are unique (for a given entropy pair). For systems, there are various choices for numerical fluxes which meet the entropy conservation requirement (3.11). In Section 5 we restricted our attention to just one such choice. In this section we present the general framework.

The entropy-conservative schemes treated in Section 5 are based on integration along a straight path in phase space. Consequently, the corresponding entropy stability analysis, for instance, Example 5.10, took into account only one dominant wave per cell. In contrast, in this section we introduce a new general family of entropy-conservative schemes which are based on different paths in phase space. This enables us to enforce entropy stability by fine-tuning the amount of numerical viscosity along each subpath carrying different intermediate waves. Moreover, the straight path integration of the entropy-conservative flux (5.2) does not admit a closed form, whereas the new family of entropy-conservative schemes enjoys an *explicit, closed-form formulation.* To this end, at each cell consisting of two neighbouring values

$\mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1}$, we let $\{\mathbf{r}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ be an *arbitrary* set of N linearly independent N-vectors, and let $\{\boldsymbol{\ell}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ denote the corresponding orthogonal set, $\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \mathbf{r}^k_{\nu+\frac{1}{2}} \rangle = \delta_{jk}$. Next, we introduce the intermediate states, $\{\mathbf{v}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$, starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by

$$\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^j_{\nu+\frac{1}{2}} + \Big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \mathbf{r}^j_{\nu+\frac{1}{2}}, \quad j = 1, 2, \ldots, N, \tag{6.1}$$

thus defining a path in phase space, connecting $\mathbf{v}_\nu$ to $\mathbf{v}_{\nu+1}$,

$$\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}^1_{\nu+\frac{1}{2}} + \sum_{j=1}^N \Big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \mathbf{r}^j_{\nu+\frac{1}{2}} = \mathbf{v}_\nu + \Delta\mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1}. \tag{6.2}$$

Since the mapping $\mathbf{u} \mapsto \mathbf{v}$ is one-to-one, the path is mirrored in the usual phase space of conservative variables, $\{\mathbf{u}^j_{\nu+\frac{1}{2}} := \mathbf{u}(\mathbf{v}^j_{\nu+\frac{1}{2}})\}_{j=1}^{N+1}$, starting with $\mathbf{u}^1_{\nu+\frac{1}{2}} = \mathbf{u}_\nu$ and ending with $\mathbf{u}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{u}_{\nu+1}$. Equipped with this notation we turn to our next result.

Theorem 6.1. The conservative scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}\Big],$$

with a numerical flux $\mathbf{g}^*_{\nu+\frac{1}{2}}$ given by

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \sum_{j=1}^N \frac{\psi(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}) - \psi(\mathbf{v}^j_{\nu+\frac{1}{2}})}{\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \rangle} \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \tag{6.3}$$

is an entropy-conservative approximation consistent with (2.5). Here, ψ is the entropy flux potential associated with the conserved entropy pair (U, F).

Remark. We note that the quantities on the right of (6.3) are well defined: consult (6.6) below.

Proof. The entropy conservation requirement (3.11) follows directly from (6.2) for

$$\begin{aligned}
\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}} \Big\rangle &= \sum_{j=1}^N \frac{\psi(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}) - \psi(\mathbf{v}^j_{\nu+\frac{1}{2}})}{\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \rangle} \Big\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \Big\rangle \\
&= \sum_{j=1}^N \psi\Big(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\Big) - \psi\Big(\mathbf{v}^j_{\nu+\frac{1}{2}}\Big) \\
&= \psi\Big(\mathbf{v}^{N+1}_{\nu+\frac{1}{2}}\Big) - \psi\Big(\mathbf{v}^1_{\nu+\frac{1}{2}}\Big) = \Delta\psi_{\nu+\frac{1}{2}}.
\end{aligned}$$

It remains to verify the consistency relation (3.7). Let

$$\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi) := \frac{1}{2}\Big(\mathbf{v}^{j}_{\nu+\frac{1}{2}} + \mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\Big) + \xi\Big\langle \boldsymbol{\ell}^{j}_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \mathbf{r}^{j}_{\nu+\frac{1}{2}}, \quad -\frac{1}{2} \le \xi \le \frac{1}{2}, \tag{6.4}$$

denote the straight subpath connecting $\mathbf{v}^{j}_{\nu+\frac{1}{2}}$ and $\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}$; then we can use (2.9) to express the ψ-potential jump between two consecutive intermediate states as

$$\begin{aligned}\psi\Big(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\Big) - \psi\Big(\mathbf{v}^{j}_{\nu+\frac{1}{2}}\Big) &= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi}\psi\Big(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \\ &= \Big\langle \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \Big\langle \boldsymbol{\ell}^{j}_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle.\end{aligned} \tag{6.5}$$

Inserting this into (6.3), we find that the entropy-conservative flux can be equivalently written as

$$\mathbf{g}^{*}_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \Big\langle \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \boldsymbol{\ell}^{j}_{\nu+\frac{1}{2}}, \tag{6.6}$$

and consistency is now obvious:

$$\mathbf{g}^{*}(\mathbf{v},\mathbf{v}) = \sum_{j=1}^{N} \Big\langle \mathbf{g}(\mathbf{v}), \mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \boldsymbol{\ell}^{j}_{\nu+\frac{1}{2}} = \mathbf{g}(\mathbf{v}). \tag{6.7}$$

□

Remark. We note that if we let $\Big\{\mathbf{r}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}\Big\}_{j=1}^{N+1}$ collapse into the same direction of $\Delta\mathbf{v}_{\nu+\frac{1}{2}}$, then the new entropy-conservative flux (6.5) collapses into the entropy-conservative flux of the 'first kind' studied earlier in Section 5.

As before, the new entropy-conservative schemes admit a viscosity form, subject to the phase space path. Considering a typical subpath factor on the right of (6.6), we integrate by parts along the lines of (5.3), to obtain

$$\begin{aligned}&\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi}(\xi)\Big\langle \mathbf{g}\Big(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi, \mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \\ &\quad = \frac{1}{2}\Big\langle \mathbf{f}\Big(\mathbf{u}^{j}_{\nu+\frac{1}{2}}\Big) + \mathbf{f}\Big(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\Big), \mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \\ &\qquad + \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi\Big\langle \mathbf{r}^{j}_{\nu+\frac{1}{2}}, B\Big(\mathbf{v}^{j+\frac{1}{2}}_{\nu+\frac{1}{2}}(\xi)\Big)\mathbf{r}^{j}_{\nu+\frac{1}{2}}\Big\rangle \Big\langle \boldsymbol{\ell}^{j}_{\nu+\frac{1}{2}}, \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle.\end{aligned}$$

This yields the following family of entropy-conservative schemes.

Corollary 6.2. Given a complete path in phase space,

$$\left\{\mathbf{u}^j_{\nu+\frac12} := \mathbf{u}\Big(\mathbf{v}^j_{\nu+\frac12}\Big)\right\}_{j=1}^{N+1},$$

associated with left and right orthogonal sets $\big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \mathbf{r}^k_{\nu+\frac12}\big\rangle = \delta_{jk}$, where $\mathbf{r}^j_{\nu+\frac12}$ is in the direction of $\mathbf{v}^{j+1}_{\nu+\frac12} - \mathbf{v}^j_{\nu+\frac12}$. Then we have the following entropy-conservative scheme:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\Bigg[&\sum_{j=1}^N \Big\langle \mathbf{f}\Big(\mathbf{u}^j_{\nu+\frac12}\Big) + \mathbf{f}\Big(\mathbf{u}^{j+1}_{\nu+\frac12}\Big), \mathbf{r}^j_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu+\frac12} \\
&- \sum_{j=1}^N \Big\langle \mathbf{f}\Big(\mathbf{u}^j_{\nu-\frac12}\Big) + \mathbf{f}\Big(\mathbf{u}^{j+1}_{\nu-\frac12}\Big), \mathbf{r}^j_{\nu-\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\Bigg] \\
+ \frac{1}{2\Delta x_\nu}\Bigg[&\sum_{j=1}^N \Big\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\Big\rangle \Big\langle \boldsymbol{\ell}^j_{\nu+\frac12}, \Delta\mathbf{v}_{\nu+\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu+\frac12} \\
&- \sum_{j=1}^N \Big\langle \mathbf{r}^j_{\nu-\frac12}, Q^{j+\frac12,*}_{\nu-\frac12}\mathbf{r}^j_{\nu-\frac12}\Big\rangle \Big\langle \boldsymbol{\ell}^j_{\nu-\frac12}, \Delta\mathbf{v}_{\nu-\frac12}\Big\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\Bigg], \\
Q^{j+\frac12,*}_{\nu+\frac12} &:= \int_{\xi=-\frac12}^{\frac12} 2\xi B\Big(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\Big)\,\mathrm{d}\xi.
\end{aligned} \tag{6.8}$$

The viscosity form of the entropy-conservative scheme outlined in Corollary 6.2 is a refinement of the entropy-conservative schemes (5.4). In particular, we can revisit the examples of entropy-stable recipes outlined in Section 5, using the two ingredients of (i) comparison with entropy-conservative schemes, and (ii) a proper choice of path in phase space. We continue with a discussion of these two ingredients.

(i) *Comparison.* We seek appropriate viscosity amplitudes, $q^{j+\frac12}_{\nu+\frac12}$, which upper-bound the amount of entropy-conservative viscosities on each subpath in phase space, $\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)$, so that (compare Corollary 5.1)

$$\Big\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\Big\rangle \le q^{j+\frac12}_{\nu+\frac12}. \tag{6.9}$$

A straightforward argument along the lines of our previous results yields the following result.

Theorem 6.3. The semi-discrete scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N}\left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right),\mathbf{r}^j_{\nu+\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac12} - \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac12}\right),\mathbf{r}^j_{\nu-\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\right]$$
$$+\frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N} q^{j+\frac12}_{\nu+\frac12}\left\langle \boldsymbol{\ell}^j_{\nu+\frac12},\Delta\mathbf{v}_{\nu+\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac12} - \sum_{j=1}^{N} q^{j+\frac12}_{\nu-\frac12}\left\langle \boldsymbol{\ell}^j_{\nu-\frac12},\Delta\mathbf{v}_{\nu-\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\right], \tag{6.10}$$

is entropy-stable if it contains more numerical viscosity than the entropy-conservative one in the sense that (6.9) holds.

(ii) *Choice of path.* The new ingredient here is the choice of a proper subpath in phase space. We demonstrate the advantage of using such a subpath in the context of second-order accurate reformulation of the conservative schemes outlined in Corollary 6.2. Let

$$\left\{\mathbf{w}^k(\mathbf{v}(\xi)) = \mathbf{w}^k\left(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\right)\right\}$$

be the orthonormal eigensystem of the symmetric $B = B\left(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\right)$,

$$B\left(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\right)\mathbf{w}^k(\mathbf{v}(\xi)) = b_k(\mathbf{v}(\xi))\mathbf{w}^k(\mathbf{v}(\xi)), \quad b_k(\mathbf{v}(\xi)) := \lambda_k\left(B\left(\mathbf{v}^{j+\frac12}_{\nu+\frac12}\right)\right).$$

Expanding $\mathbf{r}^j_{\nu+\frac12} = \sum_k \langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^j_{\nu+\frac12}\rangle \mathbf{w}^k(\mathbf{v}(\xi))$, we rewrite the amount of entropy-conservative viscosity corresponding to a typical subpath on the left of (6.9)

$$\left\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\right\rangle = \int_{\xi=-\frac12}^{\frac12} 2\xi\left\langle \mathbf{r}^j_{\nu+\frac12}, B\left(\mathbf{v}^{j+\frac12}_{\nu+\frac12}(\xi)\right)\mathbf{r}^j_{\nu+\frac12}\right\rangle \mathrm{d}\xi$$
$$= \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12} 2\xi b_k(\mathbf{v}(\xi))\left\langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^j_{\nu+\frac12}\right\rangle^2 \mathrm{d}\xi.$$

Simple upper bounds, for instance, $2\xi b_k(\mathbf{v}(\xi)) \le \sup_\xi |b_k(\mathbf{v}(\xi))|$, characterize the first-order Roe-type schemes. For second-order accuracy, we perform one more integration by parts along the lines of (5.20):

$$\begin{aligned}\Big\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\Big\rangle &\\ = \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\left(\frac14-\xi^2\right)&\Big[\Big\langle \nabla_{\mathbf{v}} b_k(\mathbf{v}(\xi)), \mathbf{r}^j_{\nu+\frac12}\Big\rangle \Big\langle \mathbf{w}^k(\mathbf{v}(\xi)), \mathbf{r}^j_{\nu+\frac12}\Big\rangle^2 \mathrm{d}\xi \\ &+ 2b_k(\mathbf{v}(\xi))\Big\langle \mathbf{r}^j_{\nu+\frac12}, \nabla_{\mathbf{v}}\mathbf{w}^k(\mathbf{v}(\xi))\mathbf{r}^j_{\nu+\frac12}\Big\rangle\Big]\,\mathrm{d}\xi. \qquad (6.11)\end{aligned}$$

Here, second-order accuracy is reflected by viscosity amplitudes of order $\mathcal{O}\big(|\Delta\mathbf{v}_{\nu+\frac12}|\big)$ along each subpath (being entropy-conservative, the amount of entropy dissipation is zero). How should we choose an appropriate subpath? To simplify matters we consider the symmetric case where the entropy and conservative variables coincide, $B(\mathbf{v}) = A(\mathbf{u})$. We let $\{\mathbf{u}^j_{\nu+\frac12}\}_{j=1}^N$ be the breakpoints along the path of (approximate) solutions to the Riemann problem. It is well known (Lax 1957) that each subpath is directed along the eigensystem of $A\big(\mathbf{u}^j_{\nu+\frac12}\big)$, that is, $\mathbf{u}^{j+1}_{\nu+\frac12} - \mathbf{u}^j_{\nu+\frac12} \sim \mathbf{r}^j_{\nu+\frac12}$, so $\mathbf{w}^k \sim \mathbf{r}^k_{\nu+\frac12}$ is the normalized eigensystem of A. With this choice, all but one of the terms on the right of (6.11) vanish to higher order (in $|\Delta\mathbf{u}_{\nu+\frac12}|$) and the leading term governing entropy dissipation is given by

$$\Big\langle \mathbf{r}^j_{\nu+\frac12}, Q^{j+\frac12,*}_{\nu+\frac12}\mathbf{r}^j_{\nu+\frac12}\Big\rangle \approx \int_{\xi=-\frac12}^{\frac12}\left(\frac14-\xi^2\right)\Big\langle \nabla_{\mathbf{u}} a_j\Big(\mathbf{u}^{j+\frac12}_{\nu+\frac12}(\xi)\Big), \mathbf{r}^j_{\nu+\frac12}\Big\rangle\,\mathrm{d}\xi.$$

The last expression captures the essence of the entropy-conservative schemes that balance between entropy dissipation along j-shocks, where

$$\Big\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac12}\Big\rangle > 0,$$

and the entropy production along j-rarefactions, where

$$\Big\langle \nabla_{\mathbf{v}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac12}\Big\rangle < 0.$$

To enforce entropy stability, we need to increase the amount of numerical viscosity. The use of different subpaths allows us to stabilize rarefactions while avoiding spurious entropy dissipation with shocks. A detailed study for the general nonsymmetric case requires lengthy calculations, and can be carried out along the lines of the Appendix. Here we note a simple entropy-stable correction by turning off the entropy production along the rarefactions, leading to viscosity amplitude, $q^{j+\frac12}_{\nu+\frac12}$, acting along the j-wave,

$$q^{j+\frac12}_{\nu+\frac12} = \int_{\xi=-\frac12}^{\frac12}\left(\frac14-\xi^2\right)\Big\langle \nabla_{\mathbf{u}} a_j\Big(\mathbf{u}^{j+\frac12}_{\nu+\frac12}(\xi)\Big), \mathbf{r}^j_{\nu+\frac12}\Big\rangle^+\,\mathrm{d}\xi. \qquad (6.12)$$

We conclude this section with the following two corollaries.

Corollary 6.4. The difference scheme (6.10), (6.12) is a second-order accurate entropy-stable approximation of (2.1). No artificial dissipation is added in shocks and, in particular, it has the desirable property of keeping the sharpness of shock profiles.

Next, we note that if the path connecting $\mathbf{u}^j_{\nu+\frac{1}{2}}$ and $\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}$ is chosen along the (approximate) Riemann solution, then the integrand on the right of (6.12) does not change sign. A simple upper bound of the entropy-conservative amplitude on the right of (6.12) along the lines of Example 4.7 yields an entropy-stable Lax–Wendroff-type viscosity

$$\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4}-\xi^2\right)\left\langle \nabla_{\mathbf{u}} a_j(\mathbf{u}(\xi)), \mathbf{r}^j_{\nu+\frac{1}{2}}\right\rangle^+ \mathrm{d}\xi \leq \frac{1}{4}\frac{\left[a_j(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}) - a_j(\mathbf{u}^j_{\nu+\frac{1}{2}})\right]^+}{\left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta v_{\nu+\frac{1}{2}}\right\rangle}. \tag{6.13}$$

This yields our next result.

Corollary 6.5. The following Lax–Wendroff-type difference scheme is a second-order accurate entropy-stable approximation of (2.1):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{2\Delta x_\nu}\Bigg[&\sum_{j=1}^N \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\right), \mathbf{r}^j_{\nu+\frac{1}{2}}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} \\ &- \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac{1}{2}}\right), \mathbf{r}^j_{\nu-\frac{1}{2}}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac{1}{2}}\Bigg] \\ + \frac{1}{8\Delta x_\nu}\Bigg[&\sum_{j=1}^N \left[a_j\left(\mathbf{u}^{j+1}_{\nu+\frac{1}{2}}\right) - a_j\left(\mathbf{u}^j_{\nu+\frac{1}{2}}\right)\right]^+ \boldsymbol{\ell}^j_{\nu+\frac{1}{2}} \\ &- \sum_{j=1}^N \left[a_j\left(\mathbf{u}^{j+1}_{\nu-\frac{1}{2}}\right) - a_j\left(\mathbf{u}^j_{\nu-\frac{1}{2}}\right)\right]^+ \boldsymbol{\ell}^j_{\nu-\frac{1}{2}}\Bigg].\end{aligned} \tag{6.14}$$

No artificial dissipation is added in shocks and in particular, it has the desirable property of keeping the sharpness of shock profiles.

7. Entropy stability of fully discrete schemes

In this section we study the time discretizations of the semi-discrete entropy-stable schemes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\left[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\right] \tag{7.1}$$

with essentially three-point numerical flux

$$\mathbf{g}_{\nu+\frac{1}{2}}(\mathbf{v}(t)) = \frac{1}{2}[\mathbf{f}(\mathbf{u}_\nu(t)) + \mathbf{f}(\mathbf{u}_{\nu+1}(t))] - \frac{1}{2}Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}. \tag{7.2}$$

According to Corollary 5.1, the semi-discrete scheme is entropy-stable if it contains more numerical viscosity than entropy-conservative schemes, namely $Q^*_{\nu+\frac{1}{2}} \leq \operatorname{Re} Q_{\nu+\frac{1}{2}}$. We recall (5.7), which allows us to measure the amount of entropy dissipation in (7.1) in terms of the dissipation matrices

$$D_{\nu+\frac{1}{2}} \equiv D_{\nu+\frac{1}{2}}(\mathbf{v}(t)) := Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}}. \tag{7.3}$$

The spatial part of (7.1) satisfies

$$\left\langle \mathbf{v}_\nu(t), \left[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\right]\right\rangle = \left[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\right] + \frac{1}{\Delta x_\nu}\mathcal{E}^{(x)}_\nu(\mathbf{v}(t)), \tag{7.4}$$

with $F_{\nu+\frac{1}{2}}$ being the entropy flux specified by (5.8) and $\mathcal{E}^{(x)}_\nu$ denoting the amount of entropy dissipation due to spatial discretization in (7.1), given by

$$\mathcal{E}^{(x)}_\nu := \frac{1}{4}\left\langle \Delta\mathbf{v}_{\nu-\frac{1}{2}}, D_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\right\rangle + \frac{1}{4}\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle \geq 0. \tag{7.5}$$

We note in passing that use of the dissipation matrix $D_{\nu+\frac{1}{2}}$ is restricted here to entropy-conservative schemes of the 'first kind' discussed in Section 5, and we can use a similar, refined argument with the entropy-conservative schemes of the 'second kind' in Section 6, leading to the corresponding generalization of the fully discrete entropy stability analysis presented below.

To discretize in time, we introduce a local time step, $t^{n+1} = t^n + \Delta t^{n+\frac{1}{2}}$. We shall use superscripts to denote dependence on the time level, for instance, $\mathbf{u}^n_\nu = \mathbf{u}(\mathbf{v}(x_\nu, t^n)), \mathbf{g}^n_{\nu+\frac{1}{2}} = \mathbf{g}_{\nu+\frac{1}{2}}(\mathbf{v}(t^n))$, *etc.* To simplify notation, we suppress the variability of the time step and grid cell width, abbreviating $\Delta t^{n+\frac{1}{2}}/\Delta x_{\nu+\frac{1}{2}} = \frac{\Delta t}{\Delta x}$. We shall study the entropy stability of the fully discrete schemes in terms of three prototype examples, which demonstrate the balance between the entropy dissipation from spatial stencil vs. the entropy dissipation/production due to the time discretization. We begin with the following.

Example 7.1. (Implicit backward Euler (BE) time discretization) We discretize (7.1) by the backward Euler scheme

$$\mathbf{u}^{n+1}_\nu = \mathbf{u}^n_\nu - \frac{\Delta t}{\Delta x}\left[\mathbf{g}_{\nu+\frac{1}{2}}(\mathbf{v}^{n+1}) - \mathbf{g}_{\nu-\frac{1}{2}}(\mathbf{v}^{n+1})\right], \quad \mathbf{v}^{n+1} = \mathbf{v}(\mathbf{u}(t^{n+1})). \tag{7.6}$$

We claim that the fully implicit time discretization in (7.6) is unconditionally entropy-stable. Indeed, implicit time discretization is responsible for additional entropy dissipation. For a quantitative measure of this statement, we invoke the identity

$$U(\mathbf{u}(\mathbf{v}_\nu^{n+1})) - U(\mathbf{u}(\mathbf{v}_\nu^{n})) \equiv \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\xi} U\Big(\mathbf{u}\Big(\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi)\Big)\Big)\,\mathrm{d}\xi$$
$$= \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big\langle \mathbf{v}_\nu^{n+\frac{1}{2}}(\xi), H\Big(\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi)\Big)\Delta\mathbf{v}_\nu^{n+\frac{1}{2}}\Big\rangle\,\mathrm{d}\xi, \tag{7.7}$$

where the following abbreviation is used:

$$\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi) = \frac{1}{2}\big(\mathbf{v}_\nu^{n+1} + \mathbf{v}_\nu^{n}\big) + \xi\Delta\mathbf{v}_\nu^{n+\frac{1}{2}}, \quad \Delta\mathbf{v}_\nu^{n+\frac{1}{2}} := \mathbf{v}_\nu^{n+1} - \mathbf{v}_\nu^{n}. \tag{7.8}$$

Rearranging the last term on the right of (7.7), we find that time discretization yields

$$\big\langle \mathbf{v}_\nu^{n+1}, \mathbf{u}_\nu^{n+1} - \mathbf{u}_\nu^{n}\big\rangle = U\big(\mathbf{u}_\nu^{n+1}\big) - U\big(\mathbf{u}_\nu^{n}\big) + \mathcal{E}_\nu^{BE}\big(\mathbf{v}^{n+\frac{1}{2}}\big), \tag{7.9}$$

where $\mathcal{E}_\nu^{BE}$ measures the entropy dissipation due to time discretization by backward Euler differencing:

$$\mathcal{E}_\nu^{BE}\big(\mathbf{v}^{n+\frac{1}{2}}\big) := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - \xi\right)\Big\langle \Delta\mathbf{v}_\nu^{n+\frac{1}{2}}, H\big(\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi)\big)\Delta\mathbf{v}_\nu^{n+\frac{1}{2}}\Big\rangle\,\mathrm{d}\xi \geq 0. \tag{7.10}$$

Returning to (7.6), we multiply by $\mathbf{v}_\nu^{n+1}$ and obtain entropy dissipation from both the spatial discretization (7.4), (7.5) and time discretization (7.9), (7.10)

$$\begin{aligned} U\big(\mathbf{u}_\nu^{n+1}\big) &- U(\mathbf{u}_\nu^{n}) + \frac{\Delta t}{\Delta x}\Big[F_{\nu+\frac{1}{2}}^{n+1} - F_{\nu-\frac{1}{2}}^{n+1}\Big] \\ &= \big\langle \mathbf{v}_\nu^{n+1}, \mathbf{u}_\nu^{n+1} - \mathbf{u}_\nu^{n}\big\rangle + \frac{\Delta t}{\Delta x}\Big\langle \mathbf{v}_\nu^{n+1}, \Big[\mathbf{g}_{\nu+\frac{1}{2}}\big(\mathbf{v}^{n+1}\big) - \mathbf{g}\big(\mathbf{v}^{n}\big)\Big]\Big\rangle \\ &\quad - \frac{\Delta t}{\Delta x}\mathcal{E}_\nu^{(x)}\big(\mathbf{v}^{n+1}\big) - \mathcal{E}_\nu^{BE}\big(\mathbf{v}^{n+\frac{1}{2}}\big) \\ &= -\frac{\Delta t}{\Delta x}\mathcal{E}_\nu^{(x)}\big(\mathbf{v}^{n+1}\big) - \mathcal{E}_\nu^{BE}\big(\mathbf{v}^{n+\frac{1}{2}}\big) \leq 0. \end{aligned} \tag{7.11}$$

Entropy stability is enhanced by fully implicit time discretization. In contrast, explicit time discretization, discussed in the next example, leads to entropy production. Thus, the entropy stability of explicit schemes hinges on a delicate balance between temporal entropy production and spatial entropy dissipation.

Example 7.2. (Explicit forward Euler (FE) time discretization) We discretize (7.1) by the forward Euler scheme

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^{n} - \frac{\Delta t}{\Delta x}\Big[\mathbf{g}_{\nu+\frac{1}{2}}(\mathbf{v}^{n}) - \mathbf{g}_{\nu-\frac{1}{2}}(\mathbf{v}^{n})\Big]. \tag{7.12}$$

Now, the identity (7.7), (7.8) can be put into the equivalent form

$$\big\langle \mathbf{v}_\nu^n, \mathbf{u}_\nu^{n+1} - \mathbf{u}_\nu^n \big\rangle = U\big(\mathbf{u}_\nu^{n+1}\big) - U\big(\mathbf{u}_\nu^n\big) - \mathcal{E}_\nu^{FE}\big(\mathbf{v}^{n+\frac{1}{2}}\big), \tag{7.13}$$

with entropy production $\mathcal{E}_\nu^{FE}(\mathbf{v}^{n+\frac{1}{2}})$ given by

$$\mathcal{E}_\nu^{FE}\big(\mathbf{v}^{n+\frac{1}{2}}\big) := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}+\xi\right) \Big\langle \Delta\mathbf{v}_\nu^{n+\frac{1}{2}}, H\big(\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi)\big)\Delta\mathbf{v}_\nu^{n+\frac{1}{2}} \Big\rangle \,\mathrm{d}\xi \geq 0. \tag{7.14}$$

We multiply (7.12) by $\mathbf{v}_\nu^n$, and together with the spatial dissipation of entropy quantified in (7.5), we arrive at

$$U\big(\mathbf{u}_\nu^{n+1}\big) - U(\mathbf{u}_\nu^n) + \frac{\Delta t}{\Delta x}\Big[F_{\nu+\frac{1}{2}}^n - F_{\nu-\frac{1}{2}}^n\Big] = -\frac{\Delta t}{\Delta x}\mathcal{E}_\nu^{(x)}(\mathbf{v}^n) + \mathcal{E}_\nu^{FE}\big(\mathbf{v}^{n+\frac{1}{2}}\big). \tag{7.15}$$

To study the entropy stability of (7.12), we therefore need to upper-bound the entropy production $\mathcal{E}_\nu^{FE}$, in terms of the spatial dissipation matrices $D_{\nu\pm\frac{1}{2}}$, which are responsible for the entropy dissipation in (7.5). We proceed as follows. From (7.14) we have

$$\begin{aligned}\mathcal{E}_\nu^{FE}\big(\mathbf{v}^{n+\frac{1}{2}}\big) &\leq \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}+\xi\right) \Big\langle \Delta\mathbf{v}_\nu^{n+\frac{1}{2}}, H\big(\mathbf{v}_\nu^{n+\frac{1}{2}}(\xi)\big)\Delta\mathbf{v}_\nu^{n+\frac{1}{2}} \Big\rangle \,\mathrm{d}\xi \\ &\leq \frac{K}{2}\big|\Delta\mathbf{v}_\nu^{n+\frac{1}{2}}\big|^2 \leq \frac{K^3}{2}\big|\Delta\mathbf{u}_\nu^{n+\frac{1}{2}}\big|^2,\end{aligned} \tag{7.16}$$

where K^2 is the condition number of H: see (5.26). To upper-bound the time differences, $\Delta\mathbf{u}_\nu^{n+\frac{1}{2}}$, we recall $\mathbf{g}(\mathbf{v}_{\nu+1}^n) - \mathbf{g}(\mathbf{v}_\nu^n) = B_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}^n$ with $B_{\nu+\frac{1}{2}}$ given in (5.32) as $B_{\nu+\frac{1}{2}} = \int B\big(\mathbf{v}_{\nu+\frac{1}{2}}^n(\xi)\big)\,\mathrm{d}\xi$. This enables us to rewrite the discrete forward Euler scheme (7.12) in the equivalent incremental form

$$\begin{aligned}\mathbf{u}_\nu^{n+1} - \mathbf{u}_\nu^n &= \frac{\Delta t}{2\Delta x}\Big[\Big(\mathbf{g}\big(\mathbf{v}_{\nu+1}^n\big) - \mathbf{g}\big(\mathbf{v}_\nu^n\big)\Big) + Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}^n \\ &\qquad + \Big(\mathbf{g}\big(\mathbf{v}_\nu^n\big) - \mathbf{g}\big(\mathbf{v}_{\nu-1}^n\big)\Big) + Q_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}^n\Big] \\ &= \frac{\Delta t}{2\Delta x}\Big[\Big(B_{\nu+\frac{1}{2}} + Q_{\nu+\frac{1}{2}}\Big)\Delta\mathbf{v}_{\nu+\frac{1}{2}}^n + \Big(B_{\nu-\frac{1}{2}} + Q_{\nu-\frac{1}{2}}\Big)\Delta\mathbf{v}_{\nu-\frac{1}{2}}^n\Big].\end{aligned}$$

Finally, we recall the viscosity matrix $Q_{\nu+\frac{1}{2}} = Q_{\nu+\frac{1}{2}}^* + D_{\nu+\frac{1}{2}}$. This enables us to rewrite the last expression as

$$\mathbf{u}_\nu^{n+1} - \mathbf{u}_\nu^n = \frac{\Delta t}{2\Delta x}\Big[\Big(\widetilde{B}_{\nu+\frac{1}{2}} + D_{\nu+\frac{1}{2}}\Big)\Delta\mathbf{v}_{\nu+\frac{1}{2}}^n + \Big(\widetilde{B}_{\nu-\frac{1}{2}} + D_{\nu-\frac{1}{2}}\Big)\Delta\mathbf{v}_{\nu-\frac{1}{2}}^n\Big], \tag{7.17}$$

where

$$\widetilde{B} := B + Q^* = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} (1+2\xi) B\Big(\mathbf{v}^n_{\nu+\frac{1}{2}}\Big)\,\mathrm{d}\xi = B_{\nu+\frac{1}{2}} + \mathcal{O}\Big(\big|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\big|\Big).$$

Squaring (7.17) we find

$$\begin{aligned}\big|\Delta \mathbf{u}_\nu^{n+\frac{1}{2}}\big|^2 \leq \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \Big[&\Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \Big(\widetilde{B}_{\nu+\frac{1}{2}} + D_{\nu+\frac{1}{2}}\Big)^2 \Delta \mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \\ &+ \Big\langle \Delta \mathbf{v}_{\nu-\frac{1}{2}}, \Big(\widetilde{B}_{\nu-\frac{1}{2}} + D_{\nu-\frac{1}{2}}\Big)^2 \Delta \mathbf{v}_{\nu-\frac{1}{2}}\Big\rangle\Big]. \end{aligned} \tag{7.18}$$

Compared with the spatial entropy dissipation in (7.5), we find that the forward Euler scheme is entropy-stable, $-\frac{\Delta t}{\Delta x}\mathcal{E}_\nu^{(x)}(\mathbf{v}^n) + \mathcal{E}_\nu^{FE}(\mathbf{v}^{n+\frac{1}{2}}) \leq 0$, provided D is sufficiently large that

$$K^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \Big(\widetilde{B}_{\nu+\frac{1}{2}} + D_{\nu+\frac{1}{2}}\Big)^2 \leq \frac{\Delta t}{\Delta x} D_{\nu+\frac{1}{2}}. \tag{7.19}$$

We consider the two prototype examples of centred and upwind schemes. If we set $D_{\nu+\frac{1}{2}} = \frac{\Delta x}{2\Delta t} I_{N\times N}$ we obtain the centred *modified Lax–Friedrichs* (MLxF) scheme (*e.g.*, Tadmor (1984*b*))

$$\mathbf{u}_\nu^{n+1} = \frac{1}{4}\big(\mathbf{u}^n_{\nu+1} + 2\mathbf{u}^n_\nu + \mathbf{u}^n_{\nu-1}\big) + \frac{\Delta t}{2\Delta x}\big[\mathbf{f}\big(\mathbf{u}^n_{\nu+1}\big) - \mathbf{f}\big(\mathbf{u}^n_{\nu-1}\big)\big]. \tag{7.20}$$

To simplify matters, we consider the symmetric case, where the Bs are turned into As, and (7.19) with condition number $K=1$ yields the entropy stability of the MLxF for sufficiently small CFL number

$$\frac{\Delta t}{\Delta x}\max_\lambda |\lambda(\widetilde{A})| \leq \frac{\sqrt{2}-1}{2}.$$

Similarly, the viscosity coefficient matrix, $D_{\nu+\frac{1}{2}} = |\widetilde{A}_{\nu+\frac{1}{2}}|$ leads to the upwind scheme

$$\begin{aligned}\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - &\frac{\Delta t}{2\Delta x}\big[\mathbf{f}\big(\mathbf{u}^n_{\nu+1}\big) - \mathbf{f}\big(\mathbf{u}^n_{\nu-1}\big)\big] \\ &+ \frac{1}{2\Delta x_\nu}\Big[\Big(\big|A_{\nu+\frac{1}{2}} + Q^*_{\nu+\frac{1}{2}}\big| + Q^*_{\nu+\frac{1}{2}}\Big)\Delta \mathbf{v}^n_{\nu+\frac{1}{2}} \\ &- \Big(\big|A_{\nu-\frac{1}{2}} + Q^*_{\nu+\frac{1}{2}}\big| + Q^*_{\nu-\frac{1}{2}}\Big)\Delta \mathbf{v}^n_{\nu-\frac{1}{2}}\Big]. \end{aligned} \tag{7.21}$$

According to (7.19), entropy stability follows under the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{\lambda,\nu}\Big|\lambda\Big(\widetilde{A}_{\nu+\frac{1}{2}}\Big)\Big| \leq 1/4, \quad \widetilde{A} := A + Q^*.$$

We conclude this fully explicit example with several remarks.

(1) *CFL optimality.* In both examples of the centred and upwind schemes, entropy stability is obtained under less than optimal CFL conditions, which is due to less than optimal bounds on the entropy production rate, $\mathcal{E}_\nu^{FE}$. In particular, the resulting entropy stability condition (7.19) excludes the entropy stability of second-order fully discrete schemes, which are identified with Lax–Wendroff (LxW) dissipation matrices of order $D_{\nu+\frac{1}{2}} \sim \frac{\Delta t}{\Delta x}\tilde{A}^2_{\nu+\frac{1}{2}}$.

(2) *Entropy stability of Lax–Wendroff scheme.* For first-order accurate schemes, sharp CFL entropy stability conditions would follow from an alternative approach discussed in Section 8 below. The question of entropy stability for second-order fully discrete schemes, however, is more delicate. It would be desirable to refine the above arguments to obtain an improved CFL condition, which in particular entertains the second-order case. For a systematic approach to enforcing entropy stability of the second-order scalar LxW scheme we refer to Majda and Osher (1978, 1979). We note the limitation that entropy stability places on fully discrete forward Euler time discretization, namely, higher-order accuracy requires spatial stencils with more than three points (Schonbek 1982). Part of the difficulty is due to lack of fully discrete entropy-conservative schemes (LeFloch and Rohde 2000, Theorem 6.1). This requires entropy production bounds of the kind discussed in the current example. Sharp entropy production bounds in the scalar case can be found in Chalons and LeFloch (2001*b*).

(3) *Entropy stability with distinguished waves.* Finally, we remark that an extension based on entropy-conservative schemes of the 'second kind' discussed in Section 6 would lead to an entropy stability statement under a refinement of the CFL statement (7.19), similar to the semi-discrete discussion in Section 6.

Example 7.3. (Crank–Nicolson time discretization) The fully explicit Euler time discretization does not conserve entropy except in the case of linear fluxes (LeFloch and Rohde 2000). Consequently, both the fully explicit and fully implicit Euler differencing do not respect (nonlinear) entropy conservation, independent of the spatial discretization. Fully discrete entropy conservation is offered by Crank–Nicolson time differencing. In its standard version, for example, Richtmyer and Morton (1967) and Gustafsson *et al.* (1995), time is replaced by divided differences centred at $t^{n+\frac{1}{2}} := \frac{1}{2}(t^n + t^{n+1})$ and spatial terms are evaluated at the mid-value, $\frac{1}{2}(\mathbf{v}^n + \mathbf{v}^{n+1})$. In the present nonlinear context, the mid-value should be weighted by the specific entropy function we are dealing with. We set

$$\overline{\mathbf{v}}^{n+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{v}\left(\frac{1}{2}\big(\mathbf{u}^n + \mathbf{u}^{n+1}\big) + \xi\Delta\mathbf{u}^{n+\frac{1}{2}}\right) \mathrm{d}\xi, \quad \Delta\mathbf{u}^{n+\frac{1}{2}} = \mathbf{u}^{n+1} - \mathbf{u}^n, \tag{7.22}$$

and we discretize (7.1) by the (generalized) Crank–Nicolson scheme

$$\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x}\Big[\mathbf{g}_{\nu+\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big) - \mathbf{g}_{\nu-\frac{1}{2}}\big(\overline{\mathbf{v}}^{n+\frac{1}{2}}\big)\Big]. \tag{7.23}$$

Noting that $\big\langle\overline{\mathbf{v}}^{n+\frac{1}{2}}, \mathbf{u}^{n+1} - \mathbf{u}^n\big\rangle = U(\mathbf{u}^{n+1}) - U(\mathbf{u}^n)$, we conclude the following.

Corollary 7.4. The Crank–Nicolson scheme (7.22), (7.23) is entropy-stable (and, respectively, entropy-conservative), if and only if the semi-discrete scheme associated with the numerical flux $\mathbf{g}(\cdot)$ is entropy-stable (respectively, entropy-conservative).

Observe that in the symmetric case, $\overline{\mathbf{v}}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}^n + \mathbf{u}^{n+1})$, and (7.23) recovers the standard differencing centred around $t^{n+\frac{1}{2}}$.

We conclude this section by referring the reader to the recent work of LeFloch and his co-workers (LeFloch and Rohde 2000, LeFloch *et al.* 2002) for a general framework along these lines for entropy stability of fully discrete schemes.

8. Entropy stability by the homotopy approach

We study the cell entropy inequality for general difference schemes written in their viscosity form corresponding to (5.24):

$$\begin{aligned}\mathbf{u}_\nu^{n+1} = \mathbf{u}_\nu^n &- \frac{\Delta t}{2\Delta x}\big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\big] \\ &+ \frac{\Delta t}{2\Delta x}\big[P_{\nu+\frac{1}{2}}\big(\mathbf{u}_{\nu+1}^n - \mathbf{u}_\nu^n\big) - P_{\nu-\frac{1}{2}}\big(\mathbf{u}_\nu^n - \mathbf{u}_{\nu-1}^n\big)\big].\end{aligned} \tag{8.1}$$

We decompose $\mathbf{u}_\nu^{n+1} = \big(\mathbf{u}_{\nu+\frac{1}{2}}^{n+1} + \mathbf{u}_{\nu-\frac{1}{2}}^{n+1}\big)/2$ where

$$\begin{aligned}\mathbf{u}_{\nu+\frac{1}{2}}^{n+1} &:= \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x}\big[\mathbf{f}\big(\mathbf{u}_{\nu+1}^n\big) - \mathbf{f}\big(\mathbf{u}_\nu^n\big)\big] + \frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\big(\mathbf{u}_{\nu+1}^n - \mathbf{u}_\nu^n\big), \\ \mathbf{u}_{\nu-\frac{1}{2}}^{n+1} &:= \mathbf{u}_\nu^n - \frac{\Delta t}{\Delta x}\big[\mathbf{f}\big(\mathbf{u}_\nu^n\big) - \mathbf{f}\big(\mathbf{u}_{\nu-1}^n\big)\big] - \frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\big(\mathbf{u}_\nu^n - \mathbf{u}_{\nu-1}^n\big),\end{aligned}$$

and we study the entropy inequality for each term. This decomposition into left- and right-handed stencils in the context of cell entropy inequality was first introduced in Tadmor (1984*b*). We begin by considering $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}$. To this end, we set $\mathbf{u}^n_{\nu+\frac{1}{2}}(s) := \mathbf{u}^n_\nu + s(\mathbf{u}^n_{\nu+1} - \mathbf{u}^n_\nu)$ and the following inequality is sought (here and below, $\Delta\mathbf{u} := \mathbf{u}^n_{\nu+1} - \mathbf{u}^n_\nu$):

$$\begin{aligned}\mathcal{I}_+ := U\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}\Big) - U\big(\mathbf{u}^n_\nu\big) + \frac{\Delta t}{\Delta x}\big[F\big(\mathbf{u}^n_{\nu+1}\big) - F\big(\mathbf{u}^n_\nu\big)\big] \\ - \frac{\Delta t}{\Delta x}\int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big), P_{\nu+\frac{1}{2}}\Delta\mathbf{u}\Big\rangle \,\mathrm{d}s \le 0. \qquad (8.2)\end{aligned}$$

We refer to the last statement as a *quasi-cell entropy inequality* since the last expression on the right is not conservative. To verify (8.2) we proceed as follows. We set

$$\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s) := \mathbf{u}^n_\nu - \frac{\Delta t}{\Delta x}\Big[\mathbf{f}\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big) - \mathbf{f}(\mathbf{u}^n_\nu)\Big] + \frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s) - \mathbf{u}^n_\nu\Big).$$

Noting that $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(0) = \mathbf{u}^n_\nu$ and $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(1) = \mathbf{u}^{n+1}_{\nu+\frac{1}{2}}$, we compute

$$\begin{aligned}&U\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}\Big) - U(\mathbf{u}^n_\nu) \\ &\quad= \int_{s=0}^1 \frac{\mathrm{d}}{\mathrm{d}s}U\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\Big)\,\mathrm{d}s \\ &\quad= \int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\Big), \Big(-\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\Big) + \frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big)\Delta\mathbf{u}\Big\rangle\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\frac{\Delta t}{\Delta x}\big[F\big(\mathbf{u}^n_{\nu+1}\big) - F\big(\mathbf{u}^n_\nu\big)\big] &= \frac{\Delta t}{\Delta x}\int_{s=0}^1 \Big\langle F'\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big), \big(\mathbf{u}^n_{\nu+1} - \mathbf{u}^n_\nu\big)\Big\rangle\,\mathrm{d}s \\ &= \frac{\Delta t}{\Delta x}\int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big)A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big), \Delta\mathbf{u}\Big\rangle\,\mathrm{d}s.\end{aligned}$$

Adding the last two equalities yields

$$\begin{aligned}\mathcal{I}_+ = \int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\Big) - U'\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big), \\ -\frac{\Delta t}{\Delta x}\Big(P_{\nu+\frac{1}{2}} - A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big)\Big)\Delta\mathbf{u}\Big\rangle\,\mathrm{d}s. \qquad (8.3)\end{aligned}$$

Next, we introduce

$$\mathbf{u}^n_{\nu+\frac{1}{2}}(r,s) := \mathbf{u}^n_{\nu+\frac{1}{2}}(s) + r\Big(\mathbf{u}^n_\nu - \mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big) \equiv \mathbf{u}^n_\nu + s(1-r)\big(\mathbf{u}^n_{\nu+1} - \mathbf{u}^n_\nu\big), \quad (8.4)$$

and we set

$$\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s)=\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)-\frac{\Delta t}{\Delta x}\Big(\mathbf{f}\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\Big)-\mathbf{f}\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\Big)\Big) \\ +\frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)-\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\Big)$$

so that $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(0,s)=\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)$ and $\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(1,s)=\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)$. This then yields

$$U'\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(s)\Big)-U'\Big(\mathbf{u}^{n}_{\nu+1}(s)\Big)=\int_{r=0}^{1}\frac{\mathrm{d}}{\mathrm{d}r}U'\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s)\Big)\,\mathrm{d}r \\ =-s\int_{r=0}^{1}U''\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s)\Big)\,\mathrm{d}r\Big(I+\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\Big)-\frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big)\boldsymbol{\Delta}\mathbf{u}.$$

Inserting the last expression into the right-hand side of (8.3) we end up with

$$\mathcal{I}_{+}=-\int_{r,s=0}^{1}s\Big\langle\Big(I+\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(r,s)\Big)-\frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big)\boldsymbol{\Delta}\mathbf{u}, \\ U''\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}(r,s)\Big)\Big(-\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}^{n}_{\nu+\frac{1}{2}}(s)\Big)+\frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}}\Big)\boldsymbol{\Delta}\mathbf{u}\Big\rangle\,\mathrm{d}r\,\mathrm{d}s. \tag{8.5}$$

To continue, we focus our attention on two prototype cases.

(i) *The scalar case.* The positivity of the last expression on the right of (8.5) follows from a CFL condition

$$\frac{\Delta t}{\Delta x}a\Big(u^{n}_{\nu+\frac{1}{2}}(s)\Big)\le\frac{\Delta t}{\Delta x}p_{\nu+\frac{1}{2}}\le I+\frac{\Delta t}{\Delta x}a\Big(u^{n}_{\nu+\frac{1}{2}}(r,s)\Big). \tag{8.6}$$

In a similar manner, the CFL condition

$$-\frac{\Delta t}{\Delta x}a\Big(u^{n}_{\nu-\frac{1}{2}}(s)\Big)\le\frac{\Delta t}{\Delta x}p_{\nu-\frac{1}{2}}\le I-\frac{\Delta t}{\Delta x}a\Big(u^{n}_{\nu-\frac{1}{2}}(r,s)\Big)$$

yields the quasi-cell entropy inequality

$$\mathcal{I}_{-}:=U\Big(u^{n+1}_{\nu-\frac{1}{2}}\Big)-U\big(u^{n}_{\nu}\big)+\frac{\Delta t}{\Delta x}\Big(F\big(u^{n}_{\nu}\big)-F\big(\mathbf{u}^{n}_{\nu-1}\big)\Big) \\ +\frac{\Delta t}{\Delta x}\int_{s=0}^{1}\Big\langle U'\Big(u^{n}_{\nu-\frac{1}{2}}(s)\Big),p_{\nu-\frac{1}{2}}\big(u^{n}_{\nu}-u^{n}_{\nu-1}\big)\Big\rangle\,\mathrm{d}s\le 0. \tag{8.7}$$

Again, the last expression is nonconservative, but together with (8.2) we end up with the cell entropy inequality.

Corollary 8.1. Consider the fully discrete scalar scheme (8.1) and assume the CFL condition

$$\frac{\Delta t}{\Delta x}\big|a\Big(u^{n}_{\nu+\frac{1}{2}}(s)\Big)\big|\le\frac{\Delta t}{\Delta x}p_{\nu+\frac{1}{2}}\le I-\frac{\Delta t}{\Delta x}\big|a\Big(u^{n}_{\nu+\frac{1}{2}}(r,s)\Big)\big| \tag{8.8}$$

is fulfilled. Then the following cell entropy inequality holds:

$$\begin{aligned}
U\big(u_\nu^{n+1}\big) &\le \frac{1}{2}\Big(U\Big(u_{\nu+\frac12}^{n+1}\Big) + U\Big(u_{\nu-\frac12}^{n+1}\Big)\Big)\\
&\le U\big(u_\nu^n\big) - \frac{\Delta t}{\Delta x}\big[F\big(u_{\nu+1}^n\big) - F\big(u_{\nu-1}^n\big)\big]\\
&\quad + \frac{\Delta t}{2\Delta x}\Bigg[\int_{s=0}^1 \Big\langle U'\Big(u_{\nu+\frac12}^n(s)\Big), p_{\nu+\frac12}\big(u_{\nu+1}^n - u_\nu^n\big)\Big\rangle\,\mathrm{d}s\\
&\qquad - \int_{s=0}^1 \Big\langle U'\Big(u_{\nu-\frac12}^n(s)\Big), p_{\nu-\frac12}\big(u_\nu^n - u_{\nu-1}^n\big)\Big\rangle\,\mathrm{d}s\Bigg].
\end{aligned}$$

Next, we extend our discussion to systems of conservation laws.

(ii) *Symmetric systems* of conservation laws with the quadratic entropy, $U(\mathbf{u}) = |\mathbf{u}|^2/2$. We start by setting $C(s) := P_{\nu+\frac12} - A(\mathbf{u}_{\nu+\frac12}^n(s))$, and noting the (r,s)-variables in (8.4), we find that

$$A\Big(\mathbf{u}_{\nu+\frac12}^n(r,s)\Big) - P_{\nu+\frac12} = -C((1-r)s)).$$

Change of variables, $t := (1-r)s$, in (8.5) then yields

$$\mathcal{I}_+ = \int_{s=0}^1 \int_{t=0}^s \Big\langle\Big(I - \frac{\Delta t}{\Delta x}C(t)\Big)\Delta\mathbf{u}, \frac{\Delta t}{\Delta x}C(s)\Delta\mathbf{u}\Big\rangle\,\mathrm{d}t\,\mathrm{d}s. \tag{8.9}$$

We now make the first requirement of positivity, assuming $C(\cdot) \ge 0$; then the positivity of $\mathcal{I}_+$ follows if and only if the corresponding eigenvalues satisfy $\lambda\big[C(s)\big(I - \frac{\Delta t}{\Delta x}C(t)\big)\big] \ge 0$. But $C(s)\big(I - \frac{\Delta t}{\Delta x}C(t)\big)$ is similar to $C^{\frac12}(s)\big(I - \frac{\Delta t}{\Delta x}C(t)\big)C^{\frac12}$, which is congruent to, and hence by Sylvester's theorem has the same number of nonnegative eigenvalues as, $I - \frac{\Delta t}{\Delta x}C(t)$. This leads to the second requirement, $\frac{\Delta t}{\Delta x}\lambda(C(\cdot)) \le 1$. Recall that $C(s) = P_{\nu+\frac12} - A\big(\mathbf{u}_{\nu+\frac12}^n(s)\big)$ is symmetric, and hence the last two requirements amount to the same CFL condition we met earlier in connection with the scalar case (8.6):

$$\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}_{\nu+\frac12}^n(s)\Big) \le \frac{\Delta t}{\Delta x}P_{\nu+\frac12} \le I + \frac{\Delta t}{\Delta x}A\Big(\mathbf{u}_{\nu+\frac12}(r,s)\Big).$$

In a similar manner, the CFL condition

$$-\frac{\Delta t}{\Delta x}A\Big(\mathbf{u}_{\nu+\frac12}^n(s)\Big) \le \frac{\Delta t}{\Delta x}P_{\nu+\frac12} \le I - \frac{\Delta t}{\Delta x}A\Big(\mathbf{u}_{\nu+\frac12}(r,s)\Big)$$

yields the quasi-cell entropy inequality for $\mathbf{u}_{\nu-\frac12}^{n+1}$, and the following conclusion.

Corollary 8.2. Consider the fully discrete scheme (8.1) consistent with the symmetric system (2.1) and assume the CFL condition[5]

$$\frac{\Delta t}{\Delta x}\left|A\left(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\right)\right| \le \frac{\Delta t}{\Delta x}P_{\nu+\frac{1}{2}} \le I - \frac{\Delta t}{\Delta x}\left|A\left(\mathbf{u}^n_{\nu+\frac{1}{2}}(r,s)\right)\right| \tag{8.10}$$

is fulfilled. Then the following cell entropy inequality holds for the quadratic entropy pair $U(\mathbf{u}) = |\mathbf{u}|^2/2, F(\mathbf{u}) = \int^{\mathbf{u}} \mathbf{f}(\mathbf{w})\,\mathrm{d}\mathbf{w} - \langle \mathbf{u}, \mathbf{f}(\mathbf{u})\rangle$:

$$\begin{aligned}
U\left(\mathbf{u}^{n+1}_\nu\right) &\le \frac{1}{2}\left(U\left(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}\right) + U\left(\mathbf{u}^{n+1}_{\nu-\frac{1}{2}}\right)\right)\\
&\le U\left(\mathbf{u}^n_\nu\right) - \frac{\Delta t}{2\Delta x}\left(F\left(\mathbf{u}^n_{\nu+1}\right) - F\left(\mathbf{u}^n_{\nu-1}\right)\right)\\
&\quad + \frac{\Delta t}{2\Delta x}\left(\int_{s=0}^1 \left\langle U'\left(\mathbf{v}^n_{\nu+\frac{1}{2}}(s)\right), P_{\nu+\frac{1}{2}}\left(\mathbf{u}^n_{\nu+1} - \mathbf{u}^n_\nu\right)\right\rangle \mathrm{d}s\right.\\
&\qquad \left. - \int_{s=0}^1 \left\langle U'\left(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\right), P_{\nu-\frac{1}{2}}\left(\mathbf{u}^n_\nu - \mathbf{u}^n_{\nu-1}\right)\right\rangle \mathrm{d}s\right).
\end{aligned}$$

We demonstrate the application of Corollaries 8.1 and 8.2 with two prototype examples of centred and upwind schemes.

Example 8.3. (Modified Lax–Friedrichs scheme) Here we set $P_{\nu+\frac{1}{2}} = \frac{\Delta x}{2\Delta t}I_{N\times N}$, leading to the *modified* Lax–Friedrichs scheme (7.20)

$$\mathbf{u}^{n+1}_\nu = \frac{1}{4}\left(\mathbf{u}^n_{\nu-1} - 2\mathbf{u}^n_\nu + \mathbf{u}^n_{\nu+1}\right) + \frac{\Delta t}{2\Delta x}\left[\mathbf{f}\left(\mathbf{u}^n_{\nu+1}\right) - \mathbf{f}\left(\mathbf{u}^n_{\nu-1}\right)\right].$$

The modified Lax–Friedrichs scheme is entropy-stable with respect to the quadratic entropy function (for symmetric systems) and for all convex entropies (for scalar equations), provided the CFL condition (8.8), (8.10) holds, which amounts to

$$\frac{\Delta t}{2\Delta x}\sup_{s,\lambda}\left|\lambda\left(A\left(\mathbf{u}_{\nu+\frac{1}{2}}(s)\right)\right)\right| \le \frac{1}{2}.$$

A linearized von Neumann stability analysis reveals that this CFL condition is sharp.

Remark. The original homotopy argument in this context of entropy stability is due to Lax (1971), where he proves the entropy stability of the Lax–Friedrichs (LxF) scheme, corresponding to $P_{\nu+\frac{1}{2}} = \frac{\Delta x}{\Delta t}I_{N\times N}$. In this

[5] We recall (consult (5.34)) that $|A|$ stands for the absolute value of A, defined by its spectral decomposition

$$|A| = R\begin{bmatrix} |a_1| & & \\ & \ddots & \\ & & |a_N| \end{bmatrix}R^{-1}.$$

special case of a two-point LxF stencil, we can apply the homotopy argument on the full stencil of the scheme. In the general case of essentially three-point schemes, (8.1), we follow the decomposition into left- and right-handed stencils and, as in Tadmor (1984*b*), this restricts the maximal viscosity coefficient to that of the modified LxF scheme. For a recent extension to entropy stability under an optimal CFL condition in the scalar case, consult Makridakis and Perthame (2003).

We now turn to discussion of entropy stability with the minimal amount of viscosity.

Example 8.4. (Upwind scheme) We set $P_{\nu+\frac{1}{2}} = p\big(A\big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\big)\big)$ with $p(\cdot)$ being any viscosity function satisfying $p(\cdot) \geq |\cdot|$: consult Example 5.6. The typical example is the upwind scheme

$$\begin{aligned}\mathbf{u}^{n+1}_\nu &= \mathbf{u}^n_\nu - \frac{\Delta t}{2\Delta x}\big[\mathbf{f}\big(\mathbf{u}^n_{\nu+1}\big) - \mathbf{f}\big(\mathbf{u}^n_{\nu-1}\big)\big] \\ &\quad + \frac{\Delta t}{2\Delta x}\Big[\Big(\sup_s \Big|A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big)\Big|\Big)\Delta\mathbf{u}^n_{\nu+\frac{1}{2}} - \Big(\sup_s \Big|A\Big(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\Big)\Big)\Big|\Delta\mathbf{u}^n_{\nu-\frac{1}{2}}\Big].\end{aligned} \tag{8.11}$$

We find that the upwind scheme is entropy-stable for the quadratic entropy function (for symmetric systems) and for all convex entropies (for scalar equations), provided the CFL condition (8.8), (8.10) holds, which amounts to

$$\frac{\Delta t}{2\Delta x}\sup_{s,\lambda}\Big|\lambda\Big(A\Big(\mathbf{u}_{\nu+\frac{1}{2}}(s)\Big)\Big)\Big| \leq 1. \tag{8.12}$$

Again, a linearized von Neumann stability analysis reveals that the CFL condition is sharp.

We conclude this section with several remarks.

(1) *Comparison with Roe scheme.* Consider the numerical viscosity of the Roe-type scheme (5.24), (5.33), $P_{\nu+\frac{1}{2}} = |\overline{A}_{\nu+\frac{1}{2}}|$. A comparison with the upwind scheme (8.11), $P_{\nu+\frac{1}{2}} = \sup_s \big|A\big(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\big)\big|$, reveals that the entropy stability of the latter is explained by taking into account *all intermediate* values between two neighbouring values, $\mathbf{u}_\nu$ and $\mathbf{u}_{\nu+1}$. An alternative entropy correction discussed in Example 5.10 adds the additional term of order $\mathcal{O}(|\Delta\mathbf{u}|)$ to compensate for the missing intermediate values.

(2) *Extensions.* The entropy stability results in this section are based on a homotopy approach. Our initial point was the essentially three-point scheme (8.1). The same homotopy approach refinement, starting with

the entropy-conservative schemes of the 'second kind' in Theorem 6.1, would yield a refinement of the above entropy stability statements. In particular, (i) the intermediate values sought in the upwind viscosities, (8.12), would be confined to each subpath, $\{\mathbf{u}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$. This leads to an entropy stability criterion which distinguishes between shock, rarefaction and contact waves; and (ii) starting the present homotopy approach with the entropy variables, rather than the entropy-conservative formulation in (8.1), enables us to extend the above entropy stability statements to the general nonsymmetric case.

(3) *Second-order accuracy.* The CFL conditions (8.8) and (8.10) are restricted to first-order accurate schemes. A similar, more careful computation along these lines enables us to treat the entropy stability of *second-order* accurate schemes. For a scalar entropy stability analysis along the lines of the second-order accurate Nessyahu–Tadmor central scheme, we refer to Nessyahu and Tadmor (1990, Appendix).

(4) *The scalar case.* More could be said on the scalar case, and we should mention a considerable amount of work in this direction. In particular, the entropy stability of fully discrete second-order schemes was systematically analysed in Osher and Tadmor (1988) (see the follow-up in Aiso (1993)). A key to enforcing entropy stability in this case is the use of *all* intermediate values within critical cells: consult Coquel and LeFloch (1995), Bouchut, Bourdarias and Perthame (1996), Johnson and Szepessy (1986) and Yang (1996c). Otherwise, entropy stability is enforced for a single entropy, as in Osher and Tadmor (1988), for example, or high-order accuracy should be given up (Yang 1996*b*).

9. Higher-order extensions

We generalize the construction of second-order entropy-conservative schemes to higher orders. To this end, we revisit the original derivation of the second-order entropy-conservative schemes (Tadmor 1986*b*), using finite element discretization. We begin with the weak formulation of the systems of conservation laws (2.6),

$$\int_\Omega \left\langle \mathbf{w}(x,t), \frac{\partial}{\partial t}\mathbf{u}(\mathbf{v}) \right\rangle = \int_\Omega \left\langle \frac{\partial}{\partial x}\mathbf{w}(x,t), \mathbf{g}(\mathbf{v}) \right\rangle \mathrm{d}x\,\mathrm{d}t, \quad \Omega \subset \mathbb{R}\times(0,T), \tag{9.1}$$

where $\mathbf{w}(\cdot)$ is an arbitrary $C_0^\infty(\Omega)$ test function. The key point is the use of the entropy variables, which enables us to use the standard finite element framework where both the primary computed solution $\mathbf{v}$ and the test function $\mathbf{w}$ belong to the same finite-dimensional scale of spaces. In particular, let the trial solution $\hat{\mathbf{v}} = \sum_\mu \mathbf{v}_\mu(t)\hat{H}_\mu(x)$ be chosen from the typical finite element space spanned by the C^0 'hat functions'

$$\hat{H}_\mu(x) = \left[\begin{array}{ll} \frac{x-x_{\mu-1}}{x_\mu - x_{\mu-1}}, & x_{\mu-1} \le x \le x_\mu, \\ \frac{x_{\mu+1}-x}{x_{\mu+1}-x_\mu}, & x_\mu \le x \le x_{\mu+1}. \end{array} \right.$$

Testing (9.1) against $\mathbf{w}(x) = w(x) = \hat{H}_\nu(x)$, the right-hand side of (9.1) yields

$$\int_{x_{\nu-1}}^{x_{\nu+1}} \frac{\partial}{\partial x}\hat{H}_\nu(x)\mathbf{g}\left(\sum_\mu \mathbf{v}_\mu(t)\hat{H}_\mu(x)\right) \mathrm{d}x\,\mathrm{d}t = \\ -\left[\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu-\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi\right], \tag{9.2}$$

where we employed a change of variables, expressed in terms of the usual $\mathbf{v}_{\nu+\frac{1}{2}} = \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}$. A second-order mass lumping on the left of (9.1) leads to

$$\int_{x_{\nu-1}}^{x_{\nu+1}} \hat{H}_\nu \frac{\partial}{\partial t}\mathbf{u}\left(\sum_\mu \mathbf{v}_\mu(t)\hat{H}_\mu(x)\right) \mathrm{d}x\,\mathrm{d}t \\ = \Delta x_\nu \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\mathbf{v}_\nu(t)) + \mathcal{O}\Big(\big|\mathbf{v}_{\nu+\frac{1}{2}}\big|\Big)^2. \tag{9.3}$$

Equating (9.2) and (9.3) while neglecting the quadratic error term, we end up with the entropy-conservative scheme (5.2):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}\Big], \quad \mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi.$$

Mass lumping preserves the entropy conservation induced by (and, in a sense, built into) the weak formulation (9.1), upon choosing $\hat{\mathbf{w}}(x,t) = \hat{\mathbf{v}}(x,t)$,

$$0 = \int_\Omega \left[\left\langle \hat{\mathbf{v}}(x,t), \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\hat{\mathbf{v}})\right\rangle - \left\langle \frac{\partial}{\partial x}\hat{\mathbf{v}}(x,t)\mathbf{g}(\hat{\mathbf{v}}(x,t))\right\rangle\right] \mathrm{d}x\,\mathrm{d}t \\ = \int_\Omega \left[\frac{\partial}{\partial t}U(\mathbf{u}(\hat{\mathbf{v}}(x,t))) + \frac{\partial}{\partial x}F(\mathbf{u}(\hat{\mathbf{v}}(x,t)))\right] \mathrm{d}x\,\mathrm{d}t. \tag{9.4}$$

Using higher-order piecewise polynomial finite element building blocks will lead to entropy-conservative schemes of any desired order. We note in passing that, with the increased order, the size of the stencil increases. For example, piecewise quadratic splines would lead to five-point entropy-conservative stencils of the following form.

Theorem 9.1. (LeFloch and Rohde 2000, Section 3) Consider the semi-discrete scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}\Big], \tag{9.5}$$

with a numerical flux, $\mathbf{g}_{\nu+\frac{1}{2}} = \mathbf{g}(\mathbf{v}_{\nu-1}, \mathbf{v}_\nu, \mathbf{v}_{\nu+1}, \mathbf{v}_{\nu+2})$, given by

$$\begin{aligned}\mathbf{g}_{\nu+\frac{1}{2}} = &\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\,\mathrm{d}\xi \\ &- \frac{1}{12}\Big[Q^{**}_{\nu+\frac{3}{2}}(\mathbf{v}_{\nu+2} - \mathbf{v}_{\nu+1}) - Q^{**}_{\nu-\frac{1}{2}}(\mathbf{v}_\nu - \mathbf{v}_{\nu-1})\Big].\end{aligned} \tag{9.6}$$

Here, $Q^{**}_{\nu+\frac{1}{2}}$ is a secondary viscosity coefficient depending on

$$Q^{**}_{\nu+\frac{1}{2}} = Q^{**}(\mathbf{v}_{\nu-1}, \mathbf{v}_\nu, \mathbf{v}_{\nu+1}) \tag{9.7}$$

The resulting five-point scheme (9.5), (9.6), (9.7) is entropy-conservative, and it is of (at least) third-order accuracy provided $Q^{**}(\mathbf{v}, \mathbf{v}, \mathbf{v}) = B(\mathbf{v})$.

We observe that the higher-order accuracy is intimately linked to the wider stencil, beyond the essentially three-point schemes discussed in Section 5. Once more, these wider stencils could serve as the starting point for an entropy stability theory for higher (than second)-order entropy-stable schemes.

REFERENCES

R. Abramov and A. Majda (2002), 'Discrete approximations with additional conserved quantities: Deterministic and statistical behavior', preprint.

R. Abramov, G. Kovačič and A. Majda (2003), 'Hamiltonian structure and statistically relevant conserved quantities for the truncated Burgers–Hopf equation', *Comm. Pure Appl. Math.* **56**, 1–46.

H. Aiso (1993), 'Admissibility of difference approximations for scalar conservation laws', *Hiroshima Math. J.* **23**, 15–61.

A. Arakawa (1966), 'Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow, Part I', *J. Comput. Phys.* **1**, 119–143.

S. Bianchini and A. Bressan (2003), 'Vanishing viscosity solutions of nonlinear hyperbolic systems', *Ann. Math.* To appear.

F. Bouchut (2002), 'Entropy satisfying flux vector splitting and kinetic BGK models', *Numer. Math.*, DOI http://dx.doi.org/10.1007/s00211-002-0426-9.

F. Bouchut, Ch. Bourdarias and B. Perthame (1996), 'A MUSCL method satisfying all the numerical entropy inequalities', *Math. Comp.* **65**, 1439–1461

A. Bressan (2003), Viscosity solutions of nonlinear hyperbolic systems, in *Hyperbolic Problems: Theory, Numerics and Applications, Proc. 9th Hyperbolic Conference* (T. Hou and E. Tadmor, eds), Springer. To appear.

C. Chalons and P. LeFloch (2001*a*), 'High-order entropy-conservative schemes and kinetic relations for van der Waals fluids', *J. Comput. Phys.* **168**, 184–206.

C. Chalons and P. LeFloch (2001*b*), 'A fully discrete scheme for diffusive-dispersive conservation law', *Numer. Math.* **89**, 493–509.

G.-Q. Chen (2000), 'Compactness methods and nonlinear hyperbolic conservation laws: Some current topics on nonlinear conservation laws', in Vol. 15 of *AMS/IP Stud. Adv. Math.*, Amer. Math. Soc., Providence, RI, pp. 33–75.

F. Coquel and P. LeFloch (1995), 'An entropy satisfying MUSCL scheme for systems of conservation laws', *CR Acad. Sci. Paris, Série 1* **320**, 1263–1268.

M. G. Crandall and A. Majda, (1980) 'Monotone difference approximations for scalar conservation laws', *Math. Comp.* **34**, 1–21.

M. G. Crandall and L. Tartar (1980), 'Some relations between non-expansive and order preserving mapping', *Proc. Amer. Math. Soc.* **78**, 385–390.

C. Dafermos (2000) *Hyperbolic Conservation Laws in Continuum Mechanics*, Springer.

P. Deift and K. T. R. McLaughlin, (1998) 'A continuum limit of the Toda lattice', *Mem. Amer. Math. Soc.* **131**, 1–216.

R. DiPerna (1979), 'Uniqueness of solutions to hyperbolic conservation laws', *Indiana University Math. J.* **28**, 137–188.

R. J. DiPerna (1983), 'Convergence of approximate solutions to conservation laws', *Arch. Rational Mech. Anal.* **82**, 27–70.

B. Engquist and S. Osher (1980), 'Stable and entropy condition satisfying approximations for transonic flow calculations', *Math. Comp.* **34**, 44–75.

L. P. Franca, I. Harari, T. J. R. Hughes, M. Mallet, F. Shakib, T. E. Spelce, F. Chalot and T. E. Tezduyar (1986), A Petrov–Galerkin finite element method for the compressible Euler and Navier–Stokes equations, in *Numerical Methods for Compressible Flows: Finite Difference, Element and Volume Techniques* (T. E. Tezduyar and T. J. R. Hughes, eds), *Amer. Soc. Mech. Eng.*, AMD, Vol. 78, pp. 19–44.

K. O. Friedrichs (1954), 'Symmetric hyperbolic linear differential equations', *Comm. Pure Appl. Math.* **7**, 345–392.

K. O. Friedrichs and P. D. Lax (1971), 'Systems of conservation laws with a convex extension', *Proc. Nat. Acad. Sci. USA* **68**, 1686–1688.

E. Godlewski and P.-A. Raviart (1996), *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer.

S. K. Godunov (1959), 'A difference scheme for numerical computation of discontinuous solutions of fluid dynamics', *Mat. Sb.* **47**, 271–306.

S. K. Godunov (1961), 'An interesting class of quasilinear systems', *Dokl. Acad. Nauk. SSSR* **139**, 521–523.

J. Goodman and P. D. Lax (1981), 'On dispersive difference schemes, I', *Comm. Pure Appl. Math.* **41**, 591–613.

B. Gustafsson, H.-O. Kreiss and J. Oliger (1995), *Time Dependent Problems and Difference Methods*, Wiley-Interscience.

A. Harten (1983*a*), 'High-resolution scheme for hyperbolic conservation laws', *J. Comput. Phys.* **49**, 357–393.

A. Harten (1983*b*), 'On the symmetric form of systems of conservation laws with entropy', *J. Comput. Phys.* **49**, 151–164.

A. Harten and J. M. Hyman (1983), 'Self-adjusting grid method for one-dimensional hyperbolic conservation laws', *J. Comput. Phys.* **50**, 235–269.

A. Harten and P. D. Lax (1981), 'A random choice finite difference scheme for hyperbolic conservation laws', *SIAM J. Numer. Anal.* **18**, 289–315.

A. Harten, J. M. Hyman, and P. D. Lax (1976), 'On finite difference approximations and entropy conditions for shocks', *Comm. Pure Appl. Math.*, **29**, 297–322.

A. Harten, B. Engquist, S. Osher, and S. Chakravarthy (1987), 'Uniformly high-order accurate essentially non-oscillatory schemes, III', *J. Comput. Phys.* **71**, 231–303.

T. Hou and P. Lax (1991), 'Dispersive approximations in fluid dynamics', *Comm. Pure Appl. Math.* **44**, 1–40.

T. J. R. Hughes, L. P. Franca, and M. Mallet (1986), 'A new finite-element formulation for computational fluid dynamics, I: Symmetric forms of the compressible Euler and Navier–Stokes equations and the second law of thermodynamics', *Comput. Methods Appl. Mech. Eng.* **54**, 223–234.

G.-S. Jiang and C. -W. Shu (1994), 'On a cell entropy inequality for discontinuous Galerkin method', *Math. Comp.* **62**, 531–538.

C. Johnson and A. Szepessy (1986), 'A shock-capturing streamline diffusion finite element method for a nonlinear hyperbolic conservation laws', Technical Report 1986-V9, Mathematics Department, Chalmers University of Technology, Goteborg.

C. Johnson, A. Szepessy, and P. Hansbo (1990), 'On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws' *Math. Comp.* **54**, 107–130.

L. Kaddouri (1993), 'Une méthode d'éléments finis discontinus pour les équations d'Euler des fluides compressibles', Thesis, Université Paris 6.

K. Khalfallah and A. Lerat (1988), 'Correction d'entropie pour des schémas numeriques approchant un système hyperbolique', *Note CR Acad. Sci.*

H.-O. Kreiss and J. Lorenz (1998), 'Stability for time-dependent differential equations', in *Acta Numerica*, Vol. 7, Cambridge University Press, pp. 203–286.

D. Króner (1997), *Numerical Schemes for Conservation Laws*, Wiley-Teubner, Stuttgart.

S. N. Kružkov (1970), 'First order quasilinear equations in several independent variables', *Math. USSR Sbornik* **10**, 217–243.

P. D. Lax (1954), 'Weak solutions of non-linear hyperbolic equations and their numerical computations', *Comm. Pure Appl. Math.* **7**, 159–193.

P. D. Lax (1957), 'Hyperbolic systems of conservation laws, II', *Comm. Pure Appl. Math.* **10**, 537–566.

P. D. Lax (1971), Shock waves and entropy, in *Contributions to Nonlinear Functional Analysis*, (E. A. Zarantonello, ed.), Academic Press, New York, pp. 603–634.

P. D. Lax (1972), *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Vol. 11 of *SIAM Regional Conference Lectures in Applied Mathematics*.

P. D. Lax (1986), 'On dispersive difference schemes', *Physica* **18D**, 250–254.

P. D. Lax and B. Wendroff (1960), 'Systems of conservation laws', *Comm. Pure Appl. Math.* **13**, 217–237.

P. D. Lax, D. Levermore and S. Venakidis (1993), The generation and propagation of oscillations in dispersive IVPs and their limiting behavior, in *Important Developments in Soliton Theory 1980–1990* (T. Fokas and V. E. Zakharov, eds), Springer, Berlin.

B. van Leer (1977), 'Towards the ultimate conservative difference scheme, III: Upstream-centered finite-difference schemes for ideal compressible flow', *J. Comput. Phys.* **23**, 263–275.

P. LeFloch (2002), *Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves*, Springer, *Lectures in Mathematics, ETH Zürich*.

P. LeFloch and C. Rohde (2000), 'High-order schemes, entropy inequalities and nonclassical shocks', *SIAM J. Numer. Anal.* **37**, 2023–2060.

P. LeFloch, J. M. Mercier and C. Rohde (2002), 'Fully discrete, entropy conservative schemes of arbitrary order', *SIAM J. Numer. Anal.* **40**, 1968–1992.

R. LeVeque (1992), *Numerical Methods for Conservation Laws*, Birkhäuser, Basel, *Lectures in Mathematics*.

R. LeVeque (2002), *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, *Texts in Applied Mathematics*.

D. Levermore and J.-G. Liu (1996), 'Large oscillations arising from a dispersive numerical scheme', *Phys. D* **99**, 191–216.

P. L. Lions and Souganidis (1995), 'Convergence of MUSCL and filtered schemes for scalar conservation laws and Hamilton–Jacobi equations', *Numer. Math.* **69**, 441–470.

A. Majda (1984), *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer, New York.

A. Majda and S. Osher (1978), 'A systematic approach for correcting nonlinear instabilities: The Lax–Wendroff scheme for scalar conservation laws', *Numer. Math.* **30**, 429–452.

A. Majda and S. Osher (1979), 'Numerical viscosity and the entropy condition', *Comm. Pure Appl. Math.* **32**, 797–838.

C. Makridakis and B. Perthame (2003), 'Sharp CFL, discrete kinetic formulation and entropy schemes for scalar conservation laws', *SIAM J. Numer. Anal.* To appear.

M. Merriam (1989), 'Towards a rigorous approach to artificial dissipation', AIAA-89-0471, Reno, Nevada.

M. S. Mock (1978), 'Some higher order difference schemes enforcing an entropy inequality', *Michigan Math. J.* **25**, 325–344.

M. S. Mock (1980), 'Systems of conservation of mixed type', *J. Diff. Eqns* **37**, 70–88.

M. Moskalkov (1980), 'Completely conservative schemes for gas dynamics', *USSR Comput. Maths. Math. Phys.* **20**, 162–170.

H. Nessyahu and E. Tadmor (1990), 'Non-oscillatory central differencing for hyperbolic conservation laws'. *J. Comput. Phys.* **87**, 408–463.

J. von Neumann and R. D. Richtmyer (1950), 'A method for the numerical calculation of hydrodynamic shocks', *J. Appl. Phys.* **21**, 232–237.

P. Olsson (1995), 'Summation by parts, projections, and stability, III', RIACS Technical report, 95.06.

S. Osher (1984), 'Riemann solvers, the entropy condition, and difference approximations', *SIAM J. Numer. Anal.* **21**, 217–235.

S. Osher (1985), 'Convergence of generalized MUSCL schemes', *SIAM J. Numer. Anal.* **22**, 947–961.

S. Osher and S. Chakravarthy (1984), 'High resolution schemes and the entropy condition', *SIAM J. Numer. Anal.* **21**, 955–984.

S. Osher and F. Solomon (1982), 'Upwind difference schemes for hyperbolic conservation laws', *Math. Comp.* **38**, 339–374.

S. Osher and E. Tadmor (1988), 'On the convergence of difference approximations to scalar conservation laws', *Math. Comp.* **50**, 19–51.

R. D. Richtmyer and K.W. Morton (1967), *Difference Methods for Initial-Value Problems*, Interscience, 2nd edn.

P. L. Roe (1981), 'Approximate Riemann solvers, parameter vectors and difference schemes', *J. Comput. Phys.* **43**, 357–372.

V. V. Rusanov (1961), 'Calculation of interaction of non-steady shock-waves with obstacles', *J. Comput. Math. Phys., USSR* **1**, 267–279.

R. Sanders (1983), 'On convergence of monotone finite difference schemes with variable spatial differencing', *Math. Comp.* **40**, 91–106.

D. Serre (1991), 'Richness and the classification of quasilinear hyperbolic systems', *Multidimensional Hyperbolic Problems and Computations*, Minneapolis, MN, *Inst. Math. Appl.*, Vol 29, Springer, pp. 315–333.

D. Serre (1999), *Systems of Conservation Laws, 1: Hyperbolicity, Entropies, Shock Waves* (English translation), Cambridge University Press.

M. Schonbek (1982), 'Convergence of solutions to nonlinear dispersive equations', *Comm. PDEs* **7**, 959–1000.

J. Smoller (1983), *Shock Waves and Reaction Diffusion Equations*, Springer, Berlin.

T. Sonar (1992), 'Entropy production in second-order three-point schemes', *Numer. Math.* **62**, 371–390.

A. Szepessy (1989), 'An existence result for scalar conservation laws using measure valued solutions', *Comm. PDEs* **14**, 1329–1350.

E. Tadmor (1984*a*), 'Skew-adjoint form for systems of conservation form', *J. Math. Anal. Appl.* **103**, 428–442.

E. Tadmor (1984*b*), 'Numerical viscosity and the entropy condition for conservative difference schemes', *Math. Comp.* **43**, 369–381.

E. Tadmor (1986*a*), 'A minimum entropy principle in the gas dynamics equations', *Appl. Numer. Math.* **2**, 211–219.

E. Tadmor (1986*b*), Entropy conservative finite element schemes, in *Numerical Methods for Compressible Flows: Finite Difference Element and Volume Techniques, Proc. Winter Annual Meeting of the Amer. Soc. Mech. Eng. AMD*, Vol. 78 (T. E. Tezduyar and T. J. R. Hughes, eds), pp. 149–158.

E. Tadmor (1987*a*), 'Entropy functions for symmetric systems of conservation laws', *J. Math. Anal. Appl.* **121**, 355–359.

E. Tadmor (1987*b*), 'The numerical viscosity of entropy stable schemes for systems of conservation laws, I', *Math. Comp.* **49**, 91–103.

E. Tadmor (1989), 'Convergence of spectral methods for nonlinear conservation laws', *SIAM J. Numer. Anal.* **26**, 30–44.

E. Tadmor (1997), Approximate solution of nonlinear conservation laws and related equations, in *Recent Advances in Partial Differential Equations and Applications, Proc. 1996 Venice Conference in Honor of Peter D. Lax and Louis Nirenberg on their 70th Birthday* (R. Spigler and S. Venakides, ed.), *AMS Proceedings Symp. Appl. Math.* Vol. 54, Providence, RI, pp. 321–368.

E. Tadmor (1998), Approximate solutions of nonlinear conservation laws, in *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Lecture Notes from CIME Course Cetraro, Italy, 1997* (A. Quarteroni, ed.), Vol. 1697 of *Lecture Notes in Mathematics*, Springer, pp. 1–150.

L. Tartar (1975), Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, Vol. 4* (R. J. Knopps, ed.), Vol. 39 of *Research Notes in Mathematics*, Pitman Press, pp. 136–211.

A. I. Vol'pert (1967), 'The spaces BV and quasilinear equations', *Math. USSR-Sb.* **2**, 225–267.

H. Yang (1996*a*), 'On wavewise entropy inequalities for high-resolution schemes, I: The semidiscrete case' *Math. Comp.* **65**, 45–67. Supplement *Math. Comp.* **65** S1–S13.

H. Yang (1996*b*), 'Convergence of Godunov type schemes' *Appl. Math. Letters* **9**, 63–67.

H. Yang (1998), 'On wavewise entropy inequalities for high-resolution schemes, II: Fully discrete MUSCL schemes with exact evolution in small time', *SIAM J. Numer. Anal.* **36**, 1–31.

Appendix: Entropy stability of Roe-type schemes

We consider Roe-type schemes of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = &-\frac{1}{2\Delta x_\nu}\big[f(\mathbf{u}_{\nu+1}) - f(\mathbf{u}_{\nu-1})\big] \\ &+\frac{1}{2\Delta x_\nu}\Big[p_{\nu+\frac12}\Delta\mathbf{u}_{\nu+\frac12} - p_{\nu-\frac12}\Delta\mathbf{u}_{\nu-\frac12}\Big], \quad p_{\nu+\frac12} = p\Big(\overline{A}_{\nu+\frac12}\Big).\end{aligned} \tag{A.1}$$

Once more (consult Example 5.2), we use (5.10) to rewrite the viscous part of (A.1) in terms of the entropy variables, obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}_\nu(t) = &-\frac{1}{2\Delta x_\nu}\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] \\ &+\frac{1}{2\Delta x_\nu}\Big[Q_{\nu+\frac12}\Delta\mathbf{v}_{\nu+\frac12} - Q_{\nu-\frac12}\Delta\mathbf{v}_{\nu-\frac12}\Big],\end{aligned} \tag{A.2}$$

where

$$Q_{\nu+\frac12} = \int_{\xi=-\frac12}^{\frac12} p\Big(\overline{A}_{\nu+\frac12}\Big)H(\xi)\,\mathrm{d}\xi, \quad H(\xi) \equiv H\Big(\mathbf{v}_{\nu+\frac12}(\xi)\Big). \tag{A.3}$$

Corollary 5.1 suggests that the entropy dissipation of these schemes should be measured by the quantity $\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \rangle$. A lower bound for the latter is provided in the following lemma.

Lemma A1. Let $\{\overline{\mathbf{r}}_k, \overline{a}_k\}$ be the eigensystem of $\overline{A}_{\nu+\frac{1}{2}}$, and assume that

$$|\overline{\mathbf{r}}_k - \mathbf{r}_k(\xi = 0)| + |\overline{a}_k - a_k(\xi = 0)| \leq \text{Const}\left|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\right|^2. \tag{A.4}$$

Then we have

$$\left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \geq \sum_{k=1}^{N} \left[p(\overline{a}_k) - \text{Const}\left|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\right|^2 \right] \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left| \left\langle \mathbf{r}_k(\xi), \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \right|^2 \mathrm{d}\xi. \tag{A.5}$$

Proof. Using the orthonormal system $\{H^{-\frac{1}{2}}(\xi)\mathbf{r}_k(\xi)\}$ in (5.18), we can expand the right-hand side of the equality (A.3), which we rewrite as

$$\left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, p\left(\overline{A}_{\nu+\frac{1}{2}}\right) H^{\frac{1}{2}}(\xi) \cdot H^{\frac{1}{2}}(\xi) \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \mathrm{d}\xi,$$

and find

$$\left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle = \sum_{k=1}^{N} \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, p\left(A_{\nu+\frac{1}{2}}\right) \mathbf{r}_k(\xi) \right\rangle \alpha_k(\xi)\, \mathrm{d}\xi, \tag{A.6}$$

where $\alpha_k(\xi)$ abbreviates $\alpha_k(\xi) := \left\langle \mathbf{r}_k(\xi), \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle$.

Consider the quantities on the right of (A.6): their dependence on ξ is reflected through their dependence on $\mathbf{v}_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi \Delta v_{\nu+\frac{1}{2}}$; for such quantities we have

$$\left| \frac{\mathrm{d}^s X(\xi)}{\mathrm{d}\xi^s} \equiv \frac{\mathrm{d}^s}{\mathrm{d}\xi^s} X\left(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\right) \right| = \mathcal{O}\left(\left|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\right|^s\right). \tag{A.7}$$

By Taylor's theorem,

$$\mathbf{r}_k(\xi) = \mathbf{r}_k(0) + \xi \dot{\mathbf{r}}_k(0) + \frac{\xi^2}{2} \ddot{\mathbf{r}}_k(\theta\xi), \quad \text{for some } \theta \in [0,1].$$

Here and below, $\dot{(\ )}$ denotes ξ-differentiation, $\frac{\mathrm{d}}{\mathrm{d}\xi}(\ )$. In view of (A.7), $|\ddot{\mathbf{r}}_k| \leq \text{Const}\left|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\right|^2$, and together with assumption (A.4) we have

$$\mathbf{r}_k(\xi) = \overline{\mathbf{r}}_k + \xi \dot{\mathbf{r}}_k(0) + J_k, \quad |J_k| \leq \text{Const}\left|\Delta \mathbf{v}_{\nu+\frac{1}{2}}\right|^2. \tag{A.8}$$

Inserting this into (A.6) we conclude that

$$\begin{aligned}
&\left\langle \Delta\mathbf{v}_{\nu+\frac12}, Q_{\nu+\frac12}\Delta\mathbf{v}_{\nu+\frac12}\right\rangle \\
&\quad = \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\left\langle \Delta\mathbf{v}_{\nu+\frac12}, p\Big(\overline{A}_{\nu+\frac12}\Big)\overline{\mathbf{r}}_k\right\rangle \alpha_k(\xi)\,\mathrm{d}\xi \\
&\qquad + \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\xi\left\langle \Delta\mathbf{v}_{\nu+\frac12}, p\Big(\overline{A}_{\nu+\frac12}\Big)\dot{\mathbf{r}}_k(0)\right\rangle \alpha_k(\xi)\,\mathrm{d}\xi \\
&\qquad + \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\left\langle \Delta\mathbf{v}_{\nu+\frac12}, p\Big(\overline{A}_{\nu+\frac12}\Big)J_k\right\rangle \alpha_k(\xi)\,\mathrm{d}\xi. \qquad \text{(A.9)}
\end{aligned}$$

Finally, taking into account (A.8), we have

$$p\Big(\overline{A}_{\nu+\frac12}\Big)\overline{\mathbf{r}}_k = p(\overline{a}_k)\mathbf{r}_k = p(\overline{a}_k)\mathbf{r}_k(\xi) - \xi p(\overline{a}_k)\dot{\mathbf{r}}_k(0) - p(\overline{a}_k)J_k;$$

we substitute the last three terms into three summations on the right of (A.9), respectively, and end up with

$$\begin{aligned}
&\left\langle \Delta\mathbf{v}_{\nu+\frac12}, Q_{\nu+\frac12}\Delta\mathbf{v}_{\nu+\frac12}\right\rangle \\
&\quad = \sum_{k=1}^{N} p(\overline{a}_k)\cdot\int_{\xi=-\frac12}^{N}\alpha_k^2(\xi)\,\mathrm{d}\xi \\
&\qquad + \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\xi\left\langle \Delta\mathbf{v}_{\nu+\frac12}, \Big[p\Big(\overline{A}_{\nu+\frac12}\Big) - p(\overline{a}_k)\cdot I_N\Big]\dot{\mathbf{r}}_k(0)\right\rangle \alpha_k(\xi)\,\mathrm{d}\xi \\
&\qquad + \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\left\langle \Delta\mathbf{v}_{\nu+\frac12}, \Big[p\Big(\overline{A}_{\nu+\frac12}\Big) - p(\overline{a}_k)\cdot I_N\Big]J_k\right\rangle \alpha_k(\xi)\,\mathrm{d}\xi \\
&\quad = \sum_{k=1}^{N} p\Big(\overline{a}_k\Big)\int_{\xi=-\frac12}^{\frac12}\left|\left\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac12}\right\rangle\right|^2\mathrm{d}\xi + II + III. \qquad \text{(A.10)}
\end{aligned}$$

Since $J_k\alpha_k(\xi)$ is of order $\mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac12}\big|^3\big)$, we have

$$|III| \le \mathrm{Const}_{III}\cdot\big|\Delta\mathbf{v}_{\nu+\frac12}\big|^4; \qquad \text{(A.11)}$$

integration by parts of the second summation on the right of (A.10) yields

$$II = \sum_{k=1}^{N}\int_{\xi=-\frac12}^{\frac12}\left(\frac14 - \xi^2\right)\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\left\langle \Delta\mathbf{v}_{\nu+\frac12}, \Big[p\Big(\overline{A}_{\nu+\frac12}\Big) - p(\overline{a}_k)I_N\Big]\cdot\mathbf{r}_k(0)\right\rangle\alpha_k(\xi)\right\}\mathrm{d}\xi.$$

Since $\dot{\mathbf{r}}_k(0)\alpha_k(\xi)$ is of order $\mathcal{O}\left(\left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^2\right)$, the expansion inside the curly brackets is $\mathcal{O}\left(\left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^3\right)$, and its ξ-derivative gives us (consult (A.7))

$$|II| \leq \mathrm{Const}_{II} \cdot \left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^4. \tag{A.12}$$

The result (A.5) now follows, noting that $H(\xi) \leq K \cdot I_N$ (see (5.26)), and hence

$$\begin{aligned}\left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^2 &\leq \frac{1}{K}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left|H^{\frac{1}{2}}(\xi)\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^2 \mathrm{d}\xi \\ &\leq \frac{1}{K}\sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left|\left\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle\right|^2 \mathrm{d}\xi;\end{aligned} \tag{A.13}$$

consequently, (A.11), (A.12), and (A.13) imply

$$\begin{aligned}&II + III \geq \\ &\quad -\sum_{k=1}^{N}\frac{1}{K}(\mathrm{Const}_{II} + \mathrm{Const}_{III}) \cdot \left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^2 \cdot \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left|\left\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle\right|^2 \mathrm{d}\xi\end{aligned}$$

and together with (A.10), this amounts to having (A.5). □

Next we turn to bounding the viscosity part of the entropy-conservative scheme (5.4), (5.5) from above, in the spirit of Lemma A1.

Lemma A2. Let $Q^*_{\nu+\frac{1}{2}}$ be the viscosity matrix associated with the entropy-conservative scheme (5.2), (5.3). Let

$$\Delta a_k(\mathbf{u}_\nu) \equiv a_k(\mathbf{u}_{\nu+1}) - a_k(\mathbf{u}_\nu)$$

denote the jump of the kth eigenvalue, $\lambda_k(A(\mathbf{u}))$, from the state on the left, $\mathbf{u}_\nu$, to its right neighbour $\mathbf{u}_{\nu+1}$. Then, for arbitrary $\varepsilon_k > 0$, we have

$$\begin{aligned}&\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle \\ &\quad \leq \sum_{k=1}^{N}\frac{1}{6}\left[\Delta a_k(\mathbf{u}_\nu) + \varepsilon_k|\overline{a}_k| + \left(1 + \frac{|\overline{a}_k|}{\varepsilon_k}\right)\mathrm{Const}\left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^2\right] \\ &\qquad \times \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left|\left\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle\right|^2 \mathrm{d}\xi.\end{aligned} \tag{A.14}$$

Proof. According to (5.19) we have

$$\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right\rangle = \sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi a_k(\xi)\alpha_k^2(\xi)\,\mathrm{d}\xi.$$

Changing variables, $\xi \to -\xi$, and averaging, we can rewrite this as

$$\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle = \sum_{k=1}^{N} \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi\big[a_k(\xi)\alpha_k^2(\xi) - a_k(-\xi)\alpha_k^2(-\xi)\big]\,\mathrm{d}\xi. \tag{A.15}$$

By Taylor's expansion,

$$\alpha_k^2(\pm\xi) = \alpha_k^2(0) \pm \xi\alpha_k(0)\dot{\alpha}_k(0) + \frac{\xi^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(\alpha_k^2(\overline{\xi})). \tag{A.16}$$

In view of (A.7), $\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(\alpha_k^2(\overline{\xi}))$ is of order $\mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\big)$, and therefore

$$\begin{aligned}&\big[a_k(\xi)\alpha_k^2(\xi) - a_k(-\xi)\alpha_k^2(-\xi)\big]\\ &\quad= \big[a_k(\xi) - a_k(-\xi)\big]\alpha_k^2(0)\\ &\qquad+ \big[a_k(\xi) + a_k(-\xi)\big]2\xi\alpha_k(0)\dot{\alpha}_k(0) + \mathcal{O}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big).\end{aligned} \tag{A.17}$$

Moreover, we have

$$\alpha_k(\pm\xi) = \alpha_k(0) \pm \xi\dot{a}_k(0) + \frac{\xi^2}{2}\ddot{a}_k, \quad |\ddot{a}_k| \leq \mathrm{Const}\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2;$$

we substitute this into the right-hand side of (A.17): since $\dot{a}_k(0), \alpha_k^2(0)$ and $\alpha_k(0)\dot{\alpha}_k(0)$ are of order $\mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|\big), \mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2\big)$, and $\mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^3\big)$ respectively, we obtain

$$\begin{aligned}&\big[a_k(\xi)\alpha_k^2(\xi) - a_k(-\xi)\alpha_k^2(-\xi)\big]\\ &\quad= 2\xi\dot{a}_k(0)\alpha_k^2(0) + 4\xi a_k(0)\alpha_k(0)\dot{\alpha}_k(0) + \mathcal{O}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big).\end{aligned} \tag{A.18}$$

Inserting the latter expression into (A.15), we find after integration that

$$\begin{aligned}&\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle\\ &\quad\leq \sum_{k=1}^{N}\frac{1}{6}\big[\dot{a}_k(0)\alpha_k^2(0) + 2a_k(0)\alpha_k(0)\dot{\alpha}_k(0)\big] + \mathrm{Const}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big).\end{aligned}$$

In view of (A.4), we can replace $a_k(0)$ by $\overline{a}_k$ and end up with

$$\begin{aligned}&\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle\\ &\quad\leq \sum_{k=1}^{N}\frac{1}{6}\big[\dot{a}_k(0)\alpha_k^2(0) + 2\overline{a}_k\alpha_k(0)\dot{\alpha}_k(0)\big] + \mathrm{Const}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big).\end{aligned} \tag{A.19}$$

We upper-bound the second term by Cauchy–Schwartz:

$$2\overline{a}_k\alpha_k(0)\dot{\alpha}_k(0) \leq \varepsilon_k|\overline{a}_k|\alpha_k^2(0) + \frac{1}{\varepsilon_k}|\overline{a}_k|(\dot{\alpha}_k(0))^2,$$

and since $(\dot{\alpha}_k(0))^2 \le \text{Const}\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4$, (A.19) gives us

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \le \sum_{k=1}^{N}\frac{1}{6}\big[\dot{a}_k(0)+\varepsilon_k|a_k|\big]\alpha_k^2(0) + \left(1+\frac{|\overline{a}_k|}{\varepsilon_k}\right)\text{Const}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big). \tag{A.20}$$

Finally, by (A.16) we have

$$\alpha_k^2(0) = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2 \,\mathrm{d}\xi + \mathcal{O}\Big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^4\Big),$$

and, according to (A.13),

$$\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2 \le \frac{1}{K}\sum_{k=1}^{N}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2 \,\mathrm{d}\xi.$$

Using this together with (A.20) implies

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \le \sum_{k=1}^{N}\frac{1}{6}\left[\dot{a}_k(0)+\varepsilon_k|\overline{a}_k| + \left(1+\frac{|\overline{a}_k|}{\varepsilon_k}\right)\text{Const}\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2\right] \times \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big|\Big\langle \mathbf{r}_k(\xi), \Delta\mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle\Big|^2 \,\mathrm{d}\xi, \tag{A.21}$$

and the result (A.14) follows, noting that $\dot{a}_k(0) = \Delta a_k(\mathbf{u}_\nu) + \mathcal{O}\big(\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2\big)$. □

We close this section with the following proof.

Proof of Theorem 5.9. Comparing (A.5) and (A.14), we conclude from Corollary 5.1 that the Roe-type scheme (A.1), (A.4) is entropy-stable provided that

$$p(\overline{a}_k) \ge \frac{1}{6}\left[\Delta a_k(\mathbf{u}_\nu) + \varepsilon_k|\overline{a}_k| + \left(1+\frac{|\overline{a}_k|}{\varepsilon_k}\right)\text{Const}\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|^2\right], \tag{A.22}$$

and since, by (5.10), (5.26),

$$\frac{1}{K}\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big| \le \big|\Delta\mathbf{u}_{\nu+\frac{1}{2}} \equiv H_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big| \le K\big|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\big|,$$

it follows that (A.22) is equivalent to (5.40). □